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AN ASYMPTOTIC ANALYSIS OF SOME NONLINEAR BOUNDARY-VALUE PROBLEMS OF CONVECTIVE MASS AND HEAT TRANSFER OF REACTING PARTICLES WITH THE FLOW

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Abstract—The first part of this paper, Section 2, considers the mass and heat transfer of an arbitrarily shaped droplet or solid particle in the translational and shear flows of an incompressible fluid in the case when the coefficient of heat conduction (diffusion) is an arbitrary function of temperature (concentration). For the translational flow three first terms of the asymptotic expansion of the mean Nusselt number in small Péclet number and two terms for the shear flow were obtained. The particles investigated were of the shape of a sphere, disk, ellipsoid and dumb-bell. The second part of the paper, Section 3, considers a simultaneous heat and mass exchange of an arbitrarily shaped particle with the translational flow of compressed gas, the determining parameters of which (the coefficients of heat conduction and diffusion, as well as the specific heat of the gas) depend arbitrarily on temperature. For the mean Nusselt and Sherwood numbers the first two terms of the corresponding asymptotic expansion in small Reynolds number have been determined. The third part of the paper, Section 4, considers the problem of mass transfer of a droplet in the case when an extractant, dissolved in it, diffuses into a continuous phase and there enters into the second-order chemical reaction with a chemisorbent. A three-term asymptotic expansion in the Péclet number has been obtained for the mean Sherwood number. The nonstationary diffusion to a reacting particle in a laminar translational flow is studied in the case when a nonisothermal chemical reaction occurs on its surface, the rate of which depends arbitrarily on temperature and concentrations.

NOMENCLATURE

A	parameter defining mean Sherwood number, equations (102) and (104)
A_n, A_m^n	arbitrary constants entering equation (20)
$A_m^{(i)}(\tau_i)$	functions entering expansion (155)
a	characteristic particle dimension (for spherical particles their radius is chosen as the characteristic dimension a), equatorial radius of the ellipsoid of revolution
a_1, a_2	radii of contacting spheres forming 'dumb-bell' particle
B	parameters, equations (102) and (104)
B_n^m	arbitrary constants entering equation (20)
$B_m^{(i)}$	unknown constants entering expansion (155)
b	polar (streamwise) radius of the ellipsoid of revolution
C_n	arbitrary constants in equation (20)
c_{e*}, c_{es}	extractant concentration in the flow and inside a droplet
c_e	dimensionless concentration of extractant, c_{e*}/c_{es}
$c_{h*}, c_{h\infty}$	chemisorbent concentration in the flow and at infinity
c_h	dimensionless concentration of chemisorbent, $c_{h*}/c_{h\infty}$
c_s	concentration on particle surface
c_*	concentration in flow
c_∞	nonperturbed concentration in the free stream at infinity
c	dimensionless concentration, $(c_\infty - c_*)/(c_\infty - c_s)$
$c_i, c^{(i)}$	terms of inner and outer expansions
$c_{m*}, c_{m\infty}$	concentration of mth reacting species in flow and at infinity
c_m	dimensionless concentration of mth reacting species, $(c_{m\infty} - c_{m*})/c_{m\infty}$
$c_{mi}, c_m^{(i)}$	terms of inner and outer expansions of reacting species concentration
c_{p*}	specific isobaric heat of gas, $c_{p*}(T_*)$
$c_p(T)$	dimensionless specific heat, $c_{p*}(T_*)/c_{p*}(T_\infty)$
D	dimensionless diffusion coefficient in equation (1), $D(c) = D_*(c_*)/D_*(c_\infty)$
D	dimensionless diffusion coefficient in equation (50), $D(T) = D_*(T_*)/D_*(T_\infty)$
D_*, D_{m*}	(dimensional) diffusion coefficients
D_{e*}, D_{h*}	diffusion coefficients of extractant and chemisorbent
E	function defined by equation (23)
$Ei(z)$	integroexponential function
$E_1(z)$	$-Ei(-z) = \int_z^\infty (e^{-\xi}/\xi)\,d\xi$
e	parameter, $(w + 1/4)\,Pe$
$\mathbf{F}$	dimensionless force of particle resistance
(F_1, F_2, F_3)	components of $\mathbf{F}$ parallel to the coordinate axes (x_1, x_2, x_3), respectively
$F_\perp, F_\parallel$	dimensionless force of the resistance of the disk (of a dumb-bell particle) with the plane (axis) normal or parallel to the free stream
$\mathbf{F}_*$	dimensional resistance force
$f(c)$	function defining equation (47)

$f_m(T,c_1,\dots,C_M)$	dimensionless rates of surface chemical reaction
$f_{m*}(T_*,c_{1*},\dots,c_{M*})$	dimensional rates of surface reaction
G_{ij}, G^*_{ij}	dimensionless and dimensional coefficients of shear matrix (58)
G	$\max_{i,j} \lvert G^*_{ij} \rvert;\ i,j = 1,2,3$
g_1, g_2	functions entering equation (19)
$g(\tau)$	function, $[2/\sqrt{(\pi\tau)}]\,e^{-\tau}+\operatorname{erf}\sqrt{\tau}$
H	function defined in equation (126), $H(c_e, c_h)$
H_m	heat of *m*th reaction
h_m	dimensionless heat of *m*th reaction, $c_{m\infty}H_mD_{m*}(\lambda_*T_\infty)^{-1}$
I_j	integrals entering equation (24); $j = 1,2,3,4,5,6$
$I_{n+1/2}$	modified Bessel function of the first kind
$\mathbf{i}$	unit vector parallel to the liquid velocity vector at infinity, $\mathbf{U}_\infty/U_\infty$
$J(\sigma)$	functional, $\int_0^1 \sigma(\xi)\,d\xi$
$K_{n+1/2}$	modified Bessel function of the second kind (McDonald function)
k_v, k_s	constant of volumetric and surface chemical reaction rate, respectively
k	dimensionless constant of volumetric and surface chemical reaction rate, $ak_sD_*^{-1}$
Le	Lewis–Semenov number, Pr/Sc
$L_n(z)$	coefficients in the expansion of the function $L(z,\eta)$ in Legendre polynomials
l	parameter, $c_{es}/c_{h\infty}$
M	number of reacting species
Nu	mean Nusselt number, $Nu(\lambda, Pe_T)$
n	order of shear flow
P_n	Legendre polynomials
P_n^m	associated Legendre functions of the first kind *n*th degree and *m*th order
Pe	diffusional Péclet number, $aU/D_*(c_\infty)$, $aU/D_*(T_\infty)$
Pe_m	diffusional Péclet numbers, aUD_{m*}^{-1}
Pe_T	thermal Péclet number, $aUc_{p*}(T_\infty)\rho_\infty\lambda_*^{-1}(T_\infty)$
Pr	Prandtl number, $\mu_*c_{p*}\lambda_*^{-1}$
p	complex Laplace–Carlson transformation parameter
p_*	pressure in the gas flow
p_∞	nonperturbed pressure in the free stream
$\mathbf{Q}^{(n)}$	liquid velocity at infinity in uniform shear flow of order *n*, $(Q_1^{(n)}, Q_2^{(n)}, Q_3^{(n)})$
$Q_i^{(n)}$	uniform *n*-order polynomials, $Q_i^{(n)}(\mathbf{r})$
q	parameter defining approximation for the asymptote to function Φ at infinity, equation (15)
q_m	parameters in expansion (150)
Re	Reynolds number, $aU\rho_*\mu_*^{-1}$
R	gas constant
$R(c_e, c_h)$	dimensionless rate of volumetric chemical reaction
$R_*(c_{e*}, c_{h*})$	dimensional rate of volumetric reaction
r	r_*/a
r_*	dimensional radial coordinate
$\mathbf{r}$	spherical coordinate system connected with the particle, (r, θ, ϕ)
r	particle surface equation, $r_s(\theta, \phi)$
r_s	represents a short notation for $r_s(\theta, \phi)$
S	particle surface, $\{r = r_s(\theta, \phi)\}$
S_j	functions in equation (20)
s	parameter, $D_hD_e^{-1}$
Sh	mean Sherwood number, $Sh(D, Pe) = -\frac{1}{4\pi}\int_S D\frac{\partial c}{\partial n}\,dS$
$Sh(D,0), Sh(1,0)$	mean Sherwood number corresponding to mass transfer of a stationary particle for the nonlinear and linear problem
Sh_0	mean Sherwood number corresponding to mass transfer between a particle and a stationary medium with a first-order chemical reaction occurring on the particle surface
Sc	Schmidt number, $\mu_*(\rho_*D_*)^{-1}$
$Sh_{m\infty}$	$Sh(1, Pe_m);\ m = 0,1,\dots,M$
T_s	surface temperature of particle
T_*	temperature in the flow
T_∞	nonperturbed temperature in the flow at infinity
T	dimensionless temperature, $(T_\infty - T_*)/(T_\infty - T_s)$
t, t_*	dimensionless and dimensional time
$\mathbf{U}$	liquid velocity vector
$\mathbf{U}_\infty$	flow velocity at infinity, $U_\infty = \lvert\mathbf{U}_\infty\rvert$
U	characteristic flow velocity
u	dimensionless relative concentration, $(u_\infty - u_*)/(u_\infty - u_s)$
u_*	relative (molar) concentration in the flow
u_s, u_∞	relative concentration on the particle surface and at infinity
$\bar{u}, \delta u$	functions being the solutions of auxiliary boundary-value problems (99) and (100)
V	control liquid volume between the particle surface, S, and the surface, Σ_r, of the sphere (of large enough radius) which encompasses the whole particle
$\mathbf{v}$	dimensionless liquid velocity vector, $\mathbf{U}U^{-1} = (v_1, v_2, v_3)$
W_n	functions entering equation (20)
w	parameter in equation (127), $(\partial H/\partial c_e)\,(0,1)$
X_n	surface spherical harmonics defined in equation (20)
(x_1, x_2, x_3)	rectangular Cartesian coordinate system connected with the particle
x_1	$r\cos\theta$
x_2	$r\sin\theta\cos\phi$
x_3	$r\sin\theta\sin\phi$

Y_j	functions defined in equation (20); $j = 0, 1, 2$
y	function defined in equation (54), $y(T)$
z	'contracted' radial coordinate in the case of translational flow, $Pe\ r$
$\mathbf{z}$	'contracted' spherical system of coordinates, (z, θ, ϕ)
z_m	auxiliary 'contracted' coordinates, $Pe_m\ r$
z_ν	'contracted' radial coordinate in the case of arbitrary shear flow (of order n), $Pe^\nu\ r$.

Greek symbols

α	numerical factor determined by expansion of the fundamental solution (64)
β	droplet and surrounding liquid viscosity ratio
$\beta(Pe), \beta(Pe, t)$	functions determined by equations (21) and (42)
Γ	adsorption equilibrium constant
$\Gamma(x)$	gamma-function, $\int_0^\infty e^{-\xi}\xi^{x-1}\,d\xi$
γ	Euler constant, 0.577215...
$\gamma(r, \eta)$	function in equation (21)
Δ	Laplace operator in the spherical coordinate system (r, θ, ϕ)
Δ_z	Laplace operator in the contracted system of coordinates (z, θ, ϕ), $\Delta\|_{r=z}$
δ	ratio between radii of particle and film envelope
δ_{ij}	Kronecker delta
ε	angle between axis of disk (dumb-bell particle) and direction of free-stream flow
ζ	fundamental solution of equation (63)
η	$(\mathbf{i}\cdot\mathbf{r})/r = \cos\theta$
Θ_m	parameters in equation (149)
κ	parameter, $(dD/dc)_{c=0}$
Λ	new auxiliary variable, $J^{-1}(D)\Phi(D, c)$
$\Lambda_i, \Lambda^{(i)}$	terms of the inner and outer expansion of function Λ $(i = 0, 1, 2)$
λ_*	dimensional heat conduction coefficient, $\lambda_*(T_*)$
$\lambda(T)$	dimensionless heat conduction coefficient, $\lambda_*(T_*)/\lambda_*(T_\infty)$
μ_*	dynamic viscosity of liquid around a particle
$\mu(T)$	$\mu_*(T_*)/\mu_*(T_\infty)$
ν	parameter, $(n+1)^{-1}$
Ξ	parameter in equation (141), $U\Gamma D^{-1}$
ρ_*, ρ_∞	density of gas (liquid) and nonperturbed gas density at infinity
ρ	dimensionless density, ρ_*/ρ_∞
Σ_r	surface of sphere (of radius r) surrounding the particle
σ	$\sigma(T) = \rho D$
$\sigma_{ij}(\mathbf{r}, c)$	functions defining equation (47)
τ	characteristic time, $\frac{1}{4}Pe\ t$
τ_m	characteristic time $(m = 0, 1, \ldots, M)$, $\frac{1}{4}Pe_m\ t$
Φ	new variable in equation (5), $\Phi(D, c) = \int_0^c D(\xi)\,d\xi$
$\Phi_i, \Phi^{(i)}$	terms of inner and outer expansions of function Φ $(i = 0, 1, 2)$
$\Phi_*^{(1)}$	first-term transform of outer expansion (after Laplace–Carlson transformation)
$\bar{\Phi}, \delta\Phi$	functions being solutions of auxiliary boundary-value problems (96) and (97)
χ	parameter in equation (34)
$\Psi(Pe), \Psi_m(Pe)$	unknown functions entering equations (62), (153) and (154)
ψ	logarithmic gamma-function derivative, $\psi(x)$
ψ_{st}	Hadamard–Rybchinsky stream-function corresponding to Stokes flow around spherical particle
Ω_1	inner region for translational flow, $\{r_s \leqslant r \leqslant O(Pe^{-1})\}$
Ω_∞	outer region for translational flow, $\{O(Pe^{-1}) \leqslant r\}$
$\Omega_1^{(\nu)}, \Omega_\infty^{(\nu)}$	inner and outer regions for arbitrary shear flow
$\omega_{ij}(\mathbf{r})$	functions defining equation (47).

Subscripts

s	parameters on particle surface
$*$	parameters in liquid or gas flow
∞	parameters at infinity.

1. INTRODUCTION

THE PROBLEM of the stationary heat and mass transfer of a solid sphere in a translational Stokes flow at small Péclet numbers was first studied by the method of matched asymptotic expansions [1]. Constant concentrations far from the particle and on its surface were assumed. The first five terms of the asymptotic expansion were obtained for the mean Sherwood number. Brenner [2] extended this problem to the case of an arbitrarily shaped particle or a droplet and obtained a three-term expansion in the Péclet number for the mean Sherwood number. Rimmer [3, 4] considered a similar problem for a sphere. For the field of liquid velocities he used Proudman and Pearson's results [5] obtained by the method of matched asymptotic expansions in small Reynolds number. A corresponding nonstationary analogue of this problem was studied by Choudhury and Drake [6].

Frankel and Acrivos [7] considered the stationary convective diffusion to a sphere freely suspended in a simple shear flow. A two-term expansion in Péclet number has been obtained for the mean Sherwood number. These results have been extended by Batchelor [8] for the case of an arbitrarily shaped particle freely suspended in an arbitrary linear flow. Using Batchelor's results Acrivos [9] has obtained four (for a sphere) and three (for an arbitrarily shaped particle) first terms of the corresponding asymptotic expansion.

The convective diffusion to a sphere and an arbitrarily shaped particle in a uniform translational flow with the isothermal first-order reaction occurring on its surface was considered in the works of Gupalo and Ryazantsev [10], Goldberg *et al.* [11] and Gupalo *et al.* [12]. Taylor [13] studied the mass exchange of a sphere with the first- and second-order chemical reactions occurring on its surface. The case for arbitrary kinetics of the surface reaction for a Stokes translational and shear flow past a sphere was studied in the works of Gupalo *et al.* [14] and Polyanin [15].

The stationary multi-component diffusion to a conducting reacting sphere in a laminar translational and shear flow with a nonisothermal chemical reaction occurring on its surface at the rate arbitrarily dependent on temperature and concentrations was investigated in the works of Polyanin [16, 17]. It was shown that in order to determine the integral heat and mass inflows of reacting species to the particle at small Péclet numbers it was sufficient to solve a universal algebraic (transcendental) system of equations which is much simpler than the initial system of partial differential equations. The case of a nonisothermal chemical surface reaction occurring by the Arrhenius law was considered in ref. [16].

Benilov *et al.* [18] studied a simultaneous stationary heat and mass transfer to the surface of a solid sphere in a translational gas flow. It was assumed that the Prandtl and Schmidt numbers and the specific heat of the gas were constant (independent of temperature), while the gas viscosity depended on temperature following the power law. A two-term asymptotic expansion in small Reynolds number has been obtained for the mean Nusselt and Sherwood numbers.

2. HEAT AND MASS EXCHANGE OF A PARTICLE WITH AN INCOMPRESSIBLE FLUID FLOW FOR THE DIFFUSION (HEAT CONDUCTION) COEFFICIENT ARBITRARILY DEPENDING ON CONCENTRATION (TEMPERATURE)

2.1. *Statement of the problem: introduction of a new variable*

Consider a stationary diffusion to a particle (droplet) of any shape in a laminar flow. The diffusion coefficient is assumed to be arbitrarily dependent on concentration. In the dimensionless variables the corresponding boundary-value problem can be written in the following form

$$(\mathbf{v}\cdot\operatorname{grad} c) = Pe^{-1}\operatorname{div}(D\operatorname{grad} c), \tag{1}$$

$$r = r_s(\theta,\phi),\quad c = 1;\quad r\to\infty,\quad c\to 0, \tag{2}$$

$$c = \frac{c_\infty - c_*}{c_\infty - c_s},\quad Pe = \frac{aU}{D_*(c_\infty)},\quad D = D(c) = \frac{D_*(c_*)}{D_*(c_\infty)},\quad D(0) = 1.$$

It is assumed that $c_s \neq c_\infty$ and $D_*(c_*) \neq 0$ at $c_* \in [\tilde c_1, \tilde c_2]$, where $\tilde c_1 = \min(c_s, c_\infty)$, $\tilde c_2 = \max(c_s, c_\infty)$.

The liquid is assumed to be incompressible

$$\operatorname{div}\mathbf{v} = 0, \tag{3}$$

and the normal velocity component v_n to be zero on the surface of a solid or liquid particle

$$r = r_s(\theta,\phi),\quad v_n = (\mathbf{v}\cdot\mathbf{n}) = 0. \tag{4}$$

Equations (1) and (2) also describe the distribution of heat in the flow of an incompressible liquid outside the particle in the case when the thermal diffusivity depends on temperature (a corresponding thermal problem will be discussed at greater length at the end of section 2.6).

To simplify the analysis, the concentration c is replaced by a new auxiliary function

$$\Phi = \Phi(D,c) = \int_0^c D(\xi)\,\mathrm{d}\xi\quad (\Phi(D,0) = 0,\quad \Phi'_c(D,0) = 1). \tag{5}$$

The initial boundary-value problem [equations (1) and (2)] is reformulated in terms of the function Φ as

$$\begin{gathered}(\mathbf{v}\cdot\operatorname{grad} c(\Phi)) = Pe^{-1}\,\Delta\Phi;\\ r = r_s,\quad \Phi = J(D);\quad r\to\infty,\quad \Phi\to 0,\end{gathered} \tag{6}$$

where the function $c = c(\Phi)$ is determined by the function inversion [equation (5)] (in the linear case, $D = 1, c = \Phi$), while the parameter $J(D)$ is prescribed by the expression

$$J(D) = \int_0^1 D(\xi)\,\mathrm{d}\xi. \tag{7}$$

The main aim of Section 2 is to determine the mean Sherwood number which is the major characteristic of the intensity of mass transfer to a particle

$$Sh = Sh(D,Pe) = -\frac{1}{4\pi}\int_S D(c)\frac{\partial c}{\partial n}\,\mathrm{d}S = -\frac{1}{4\pi}\int_S \frac{\partial\Phi}{\partial n}\,\mathrm{d}S. \tag{8}$$

Here $\partial/\partial n$ is the derivative with respect to the external normal to the particle surface $S = \{r = r_s(\theta, \phi)\}$. Further it is assumed that the Péclet number is small, $Pe < O(1)$.

2.2. *A uniform translational flow: a general formula for the mean Sherwood number*

For a translational Stokes flow past an arbitrarily shaped droplet or solid particle, the liquid velocity distribution far from it is determined by the expression [19]

$$\mathbf{v} = \mathbf{i} - \frac{3}{4r}\left[\mathbf{F} + \frac{1}{r^2}(\mathbf{F}\cdot\mathbf{r})\mathbf{r}\right] + O(r^{-2}), \tag{9}$$

$$U = U_\infty = |\mathbf{U}_\infty|, \quad \mathbf{i} = \frac{\mathbf{U}_\infty}{U_\infty}, \quad \mathbf{F} = \frac{\mathbf{F}_*}{6\pi\mu_* a U_\infty}.$$

The solution of the problem [equations (6), (7) and (9)] is sought by the method of matched asymptotic expansions in the small Péclet number [1, 2–6, 10–14]. In this case the entire flow region is divided into two subregions:

the inner $\qquad \Omega_1 = \{r_s \leqslant r \leqslant O(Pe^{-1})\},$

and

the outer $\qquad \Omega_\infty = \{O(Pe^{-1}) \leqslant r\}.$

As usual, a 'contracted' coordinate, $z = Pe\, r$, is introduced in the outer region and the solution in each of the subregions is sought separately in the form of the inner expansion

$$\begin{aligned} \Phi &= \Phi_0 + Pe\,\Phi_1 + Pe^2 \ln Pe\,\Phi_2 + O(Pe^2), \quad r_s \leqslant r \leqslant O(Pe^{-1}), \\ \Phi_i &= \Phi_i(r, \theta, \phi); \quad i = 0, 1, 2, \end{aligned} \tag{10}$$

and the outer expansion

$$\begin{aligned} \Phi &= \Phi^{(0)} + Pe\,\Phi^{(1)} + Pe^2\,\Phi^{(2)} + o(Pe^2), \quad O(Pe^{-1}) \leqslant r, \\ \Phi^{(i)} &= \Phi^{(i)}(z, \theta, \phi); \quad z = Pe\, r; \quad i = 1, 2, \end{aligned} \tag{11}$$

where the zero term of the outer expansion is identically zero, $\Phi^{(0)} = 0$.

In constructing the asymptotic solution in the inner region Ω_1, the boundary condition on the particle surface is used, while in the outer region Ω_∞, the boundary condition at infinity (6); the unknown constants turning up in the course of the solution are determined by using the matching technique.

The introduction of the new variable, equation (5), produces the result that all the terms of the inner, Φ_i, and the outer, $\Phi^{(i)}$, asymptotic expansions [of the initially nonlinear boundary-value problem, equations (1) and (2)] satisfy the linear equations simultaneously (see below) in the inner Ω_1 and outer Ω_∞ regions. This yields one and the same structure of expansions (10) and (11) of the linear ($D = 1$) [2] and nonlinear (6) [$D = D(c)$ in equations (1) and (2)] problems.

By substituting equation (10) for the function Φ (in the region Ω_1) into equation (6), the terms of the inner expansion are determined by the solution of the following linear equations with the boundary conditions on the particle surface

$$\Delta\Phi_0 = 0; \quad r = r_s, \quad \Phi_0 = J(D), \tag{12}$$

$$\Delta\Phi_1 = (\mathbf{v}\cdot\text{grad}\; c(\Phi_0)); \quad r = r_s, \quad \Phi_1 = 0, \tag{13}$$

$$\Delta\Phi_2 = 0; \quad r = r_s, \quad \Phi_2 = 0. \tag{14}$$

By properly choosing the zero point of the radius vector $\mathbf{r}$, the expression for Φ_0, as for the harmonic function tending to zero as $r \to \infty$ (the condition of matching with the zero term of the outer expansion $\Phi^{(0)} = 0$) can be written in the form [2]

$$\Phi_0 = qr^{-1} + O(r^{-3}), \tag{15}$$

where q is a certain constant to be determined later on. It should be emphasized that it is not essential that the above zero point for the radius vector should be actually determined here, it is only important to know that this point is available. Note that the point may turn out to be outside the particle.

It can be shown that the constant quantity q occurring in equation (13) is associated with the Sherwood number $Sh(D, 0)$ corresponding to the mass exchange of a particle with a stationary medium (at $Pe = 0$) by the relationship

$$q = Sh(D, 0). \tag{16}$$

In fact, if the Ostrogradsky–Gauss theorem is applied to the integral of the function $\Delta\Phi_0$ taken over the control liquid volume V bounded by the particle surface S and the surface Σ_r of the sphere which contains the particle; the sphere radius r can be as large as one wishes. When determining $Sh(D, 0)$, the use of the equality $\Delta\Phi_0 = 0$, equation (12), allows the integration to be carried out, according to equation (8), over the surface Σ_r. Since $\mathrm{d}\Sigma_r = O(r^2)$, then, to calculate $Sh(D, 0)$, one can apply equation (15). This yields the required equation (16).

In the outer region Ω_∞, the principal term in the asymptotic expansion (11) of the concentration field is described, by virtue of the properties of function Φ, equation (5), by the following linear homogeneous equation

$$\Delta_z\Phi^{(1)} - (\mathbf{i}\cdot\mathrm{grad}_z\,\Phi^{(1)}) = 0; \quad z \to \infty, \quad \Phi^{(1)} \to 0, \tag{17}$$

which coincides exactly with a similar equation for the linear case $D = 1$ [2]. Therefore, the solution of equation (17), which satisfies the condition of matching with the zero term of the inner expansion (15), has the form

$$\Phi^{(1)} = \frac{q}{z}\exp\left[\frac{1}{2}z(\eta - 1)\right], \quad \eta = \frac{(\mathbf{i}\cdot\mathbf{r})}{r} = \cos\theta. \tag{18}$$

The second term of the outer expansion (11) is determined by solving the following linear inhomogeneous equation

$$\begin{gathered}\Delta_z\Phi^{(2)} - (\mathbf{i}\cdot\mathrm{grad}\,\Phi^{(2)}) = g_1 + \kappa g_2; \quad z \to \infty, \quad \Phi^{(2)} \to 0,\\ g_1 = g_1(z, \eta, \phi) = -\frac{3}{4z}\left\{\left[\mathbf{F} + \frac{1}{z^2}(\mathbf{F}\cdot\mathbf{z})\mathbf{z}\right]\cdot\mathrm{grad}_z\,\Phi^{(1)}\right\},\\ g_2 = g_2(z, \eta) = -\Phi^{(1)}(\mathbf{i}\cdot\mathrm{grad}_z\,\Phi^{(1)}), \quad \kappa = (\mathrm{d}D/\mathrm{d}c)_{c=0}.\end{gathered} \tag{19}$$

At $\kappa = 0$, equation (19) coincides with the corresponding equation for the linear case. The general solution of the inhomogeneous equation (19) is determined by the sum of three terms: the general solution of the homogeneous equation (19) at $g_1 = g_2 = 0$ [i.e. equation (17)] and partial, damping at infinity, solutions of the inhomogeneous equation (19) at $g_1 \neq 0, g_2 = 0$ and $g_1 = 0, g_2 \neq 0$. The first two terms of this sum have been obtained in ref. [2].

Omitting the intermediate calculations for $\Phi^{(2)}$, gives

$$\begin{aligned}\Phi^{(2)} = \exp\left(\frac{1}{2}z\eta\right)\Biggl\{&\left(\frac{\pi}{z}\right)^{1/2}\sum_{n=0}^{\infty}K_{n+1/2}\left(\frac{z}{2}\right)X_n(\eta, \phi)\\ &+ q\exp\left(-\frac{z}{2}\right)\sum_{j=0}^{2}S_j(z)Y_j(\eta, \phi) + \frac{\kappa q^2}{\sqrt{z}}\sum_{n=0}^{\infty}W_n(z)P_n(\eta)\Biggr\},\end{aligned}$$

$$X_n(\eta, \phi) = A_nP_n(\eta) + \sum_{m=1}^{n}(A_n^m\sin m\phi + B_n^m\cos m\phi)P_n^m(\eta),$$

$$S_0(z) = \frac{1}{2z}[\mathrm{e}^z\,E_1(z) + \ln z], \quad E_1(z) = -Ei(-z) = \int_z^\infty \frac{\mathrm{e}^{-\xi}}{\xi}\,\mathrm{d}\xi,$$

$$S_1(z) = \frac{3}{4z}\left(1 - \frac{2}{z}\right)\mathrm{e}^z\,E_1(z) - \frac{3}{4z}\left(1 + \frac{2}{z}\right)\ln z - \frac{3}{2z^2},$$

$$S_2(z) = \frac{1}{4z}\left(1 - \frac{6}{z} + \frac{12}{z^2}\right)\mathrm{e}^z\,E_1(z) + \frac{1}{4z}\left(1 + \frac{6}{z} + \frac{12}{z^2}\right)\ln z + \frac{3}{2z^2}\left(1 + \frac{6}{z}\right),$$

$$Y_0 = (\mathbf{F}\cdot\mathbf{i}) = F_1P_0(\eta), \quad Y_1 = \frac{(\mathbf{F}\cdot\mathbf{z})}{z}, \quad Y_2 = \frac{3}{2z}(\mathbf{i}\cdot\mathbf{z})Y_1 - \frac{1}{2}Y_0,$$

$$Y_k = F_1P_k(\eta) + k^{-1}(F_1\cos\phi + F_2\sin\phi)P_k^1(\eta); \quad k = 1, 2,$$

$$W_0(z) = z^{-1/2}\exp\left(-\frac{z}{2}\right)\left[-\frac{1}{z} + E_1(z) + \frac{\mathrm{e}^{-z}}{z} - \mathrm{e}^{-z}\,E_1(2z)\right],$$

$$\begin{aligned}W_n(z) = {}&K_{n+1/2}\left(\frac{z}{2}\right)\int_z^{C_n}I_{n+1/2}\left(\frac{\xi}{2}\right)L_n(\xi)\,\mathrm{d}\xi\\ &-I_{n+1/2}\left(\frac{z}{2}\right)\int_z^{\infty}K_{n+1/2}\left(\frac{\xi}{2}\right)L_n(\xi)\,\mathrm{d}\xi; \quad n = 1, 2, \ldots,\end{aligned}$$

$$L_n(z) = \frac{2n+1}{2}\int_{-1}^{1}L(z, \eta)P_n(\eta)\,\mathrm{d}\eta,$$

$$L(z, \eta) = z^{3/2}\exp\left(-\frac{1}{2}z\eta\right)g_2(z, \eta) = \frac{1}{2}z^{-1/2}\exp\left(\frac{1}{2}z\eta\right)\left[-1 + \left(1 + \frac{2}{z}\right)\eta\right]. \tag{20}$$

The constants A_n, A_n^m, B_n^m, and C_n are determined from the matching condition.

By letting z approach zero in equation (20) and taking into account that as $z \to 0$, $E_1(z) \to -\ln z - \gamma + O(z)$ ($\gamma = 0.577215\ldots$ is the Euler constant), the general solution of equation (19) as $z \to 0$ is found to contain the logarithmic term $-\frac{1}{2}q(\mathbf{F}\cdot\mathbf{i}) \ln z$ [resulting from the expansion of the first term in the second sum of equation (20)], which alters the power series nature of the inner expansion and produces a factor $Pe^2 \ln Pe$ before the third term of equation (10). Taking this into account and using equations (18) and (20), the function Φ, equation (10), on the general boundary between the outer and inner regions (at $r \to \infty$) may be presented as

$$\begin{aligned} \Phi &= \beta(Pe) + \tfrac{1}{2}Pe\, q\eta + O(Pe^2) + O(r^{-1}), \\ \beta(Pe) &= -\tfrac{1}{2}q[Pe + Pe^2 \ln Pe(\mathbf{F}\cdot\mathbf{i})]. \end{aligned} \tag{21}$$

It should be noted that in order to obtain equation (21) there is no need to predetermine the first term Φ_1 of the inner expansion (10).

Let equations (12)–(14) be multiplied by Φ_0, $Pe\,\Phi_0$, and $Pe^2 \ln Pe\,\Phi_0$, respectively, and make the addition. As a result

$$\Phi_0 \Delta\Phi = Pe\,\Phi_0(\mathbf{v}\cdot \operatorname{grad} c(\Phi_0)), \tag{22}$$

where Φ is determined by expansion (10).

Now, integrate the equality (22), taking into account the following identities

$$\begin{aligned} \Phi_0\Delta\Phi &\equiv \operatorname{div}(\Phi_0 \operatorname{grad} \Phi) - \operatorname{div}(\Phi \operatorname{grad} \Phi_0) + \Phi\Delta\Phi_0, \\ \Phi_0(\mathbf{v}\cdot \operatorname{grad} c(\Phi_0)) &\equiv \operatorname{div}(\mathbf{v}E) - E \operatorname{div} \mathbf{v}, \\ E = E(\Phi_0) &= \int_0^{c(\Phi_0)} \Phi(D,\xi)\,\mathrm{d}\xi, \end{aligned} \tag{23}$$

over the control liquid volume V, bounded by the particle surface $S = \{r = r_s(\theta, \phi)\}$ and the surface Σ_r of the sphere containing the particle. Because of the harmonicity of function Φ_0, equation (12), and the incompressibility of the liquid, equation (3), the last terms on the RHS of equation (23) are equal to zero, therefore, using the Ostrogradsky–Gauss formula, one obtains

$$\begin{aligned} &\sum_{j=1}^{6} I_j = 0, \quad I_1 = -\int_S \Phi_0 \frac{\partial\Phi}{\partial n}\,\mathrm{d}S, \quad I_2 = \int_S \Phi\frac{\partial\Phi_0}{\partial n}\,\mathrm{d}S, \\ &I_3 = Pe\int_S E(\Phi_0)(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}S, \quad I_4 = \int_{\Sigma_r} \Phi_0\frac{\partial\Phi}{\partial n}\,\mathrm{d}\Sigma_r, \\ &I_5 = -\int_{\Sigma_r} \Phi\frac{\partial\Phi_0}{\partial n}\,\mathrm{d}\Sigma_r, \quad I_6 = -Pe\int_{\Sigma_r} E(\Phi_0)(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}\Sigma_r. \end{aligned} \tag{24}$$

The first three integrals of equation (24) are calculated with account for the boundary conditions on the particle surface, equations (6) and (12), the mean Sherwood number, equation (8), and the condition that the liquid does not penetrate through the particle surface, equation (4). The calculation of the last three integrals is based on equations (9), (15) and (21) for the functions $\mathbf{v}$, Φ_0 and Φ and the expressions $\mathrm{d}\Sigma_r = O(r^2)\,\mathrm{d}\theta\,\mathrm{d}\phi$, $E(\Phi_0) = \frac{1}{2}\Phi_0^2 + o(r^{-2})$ which are valid at large values of r. Taking into account that the radius of the sphere Σ_r can be arbitrarily large, one finally obtains

$$\begin{aligned} I_1 &= 4\pi J(D)Sh(D,Pe), \quad I_2 = -4\pi J(D)Sh(D,0), \\ I_3 &= I_4 = I_6 = 0, \qquad I_5 = 4\pi q\beta(Pe). \end{aligned} \tag{25}$$

Equalities (16), (24) and (25) yield the following formula for the mean Sherwood number

$$Sh(D,Pe) = Sh(D,0) + \tfrac{1}{2}J^{-1}(D)Sh^2(D,0)\{Pe + Pe^2 \ln Pe(\mathbf{F}\cdot\mathbf{i})\} + O(Pe^2). \tag{26}$$

It is not difficult to show that in the case of mass transfer of a particle in a stationary medium [at $Pe = 0$ in equation (1)] there is a simple relationship between the mean Sherwood numbers, which correspond to the nonlinear, $D = D(c)$, and linear, $D = 1$, problems, namely

$$Sh(D,0) = J(D)Sh(1,0), \tag{27}$$

which can be proved by introducing a new variable $\hat{\Phi} = J^{-1}(D)\Phi$ in equation (12) and subsequent comparison with the solution of the problem, equations (1) and (2), at $Pe = 0$, $D = 1$.

Substituting equation (27) into the RHS of expression (26) and taking into account equation (5), one obtains

$$Sh(D,Pe) = J(D)Sh(1,Pe), \quad J(D) = \int_0^1 D(\xi)\,\mathrm{d}\xi, \tag{28}$$

where the quantity $Sh(1, Pe)$ is determined by substituting $D = 1$ into equation (26), with account for the equality $J(1) = 1$, and has the form

$$Sh(1, Pe) = Sh(1,0) + \tfrac{1}{2}Sh^2(1,0)\{Pe + Pe^2 \ln Pe(\mathbf{F}\cdot\mathbf{i})\} + O(Pe^2). \tag{29}$$

Formulae (28) and (29) extend Brenner's results [2] to the nonlinear case.

The functional $J(D)$, entering into formula (28), has the following form for several typical cases (see Section 3.5)

$$\begin{aligned} &D(c) = [1+(\kappa-1)c]^n, && J(D) = \frac{1}{n+1}\,\frac{1-\kappa^{n+1}}{1-\kappa} \quad (\kappa > 0,\ n \neq 1),\\ &D(c) = \exp(\kappa c), && J(D) = \kappa^{-1}(e^{\kappa}-1),\\ &D(c) = (1+\kappa c)^{-1}, && J(D) = \kappa^{-1}\ln(1+\kappa) \quad (\kappa > -1). \end{aligned} \tag{30}$$

The first (at $n = 1$) and the second of these expressions correspond to the linear and exponential dependence of the diffusion (heat conduction) coefficient on concentration (temperature); the third relation can be encountered in a number of adsorption problems. It is seen from formulae (28)–(30) that the presence of nonlinearity in the problem, equations (1) and (2), can substantially influence the intensity of mass exchange between the particle and the flow.

2.3. *Calculation of the mean Sherwood number for some specific cases*

For a particle of the prescribed shape the problem of determining the convective mass transfer intensity coefficient after calculation of the integral $J(D)$ by equations (28) and (29) is reduced to the determination of the number $Sh(1,0)$ and the particle resistance force $(\mathbf{F}\cdot\mathbf{i})$. Since the latter depends on the particle orientation, then, as follows from formula (29), the mean Sherwood number (28) also depends on the particle orientation in the flow. The quantity $Sh(1,0)$ is calculated by solving the conventional Laplace equation $\Delta c = 0$ subject to the boundary conditions, equation (2), and is well known for a variety of cases. Thus, for a droplet or a spherical particle $Sh(1,0) = 1$.

In what follows, some specific values of the determining parameters of the problem, $Sh(1,0)$ and $(\mathbf{F}\cdot\mathbf{i})$, will be presented for different shapes and types of the particles, as well as their orientation in the translational Stokes flow.

2.3.1. *Spherical droplet.* In this case one obtains

$$Sh(1,0) = 1, \quad (\mathbf{F}\cdot\mathbf{i}) = \frac{2+3\beta}{3+3\beta}, \tag{31}$$

where $\beta = \infty$ corresponds to a solid sphere, and $\beta = 0$, to a gas bubble.

2.3.2. *Spherical particle covered with a liquid film.* The determining parameters are of the form [20]

$$Sh(1,0) = 1, \quad (\mathbf{F}\cdot\mathbf{i}) = \frac{2}{3} + \frac{1}{3}\left[1 + \frac{1}{\beta}\,\frac{1-\delta}{1+\delta}\left(1 + \frac{5}{2}\,\frac{\delta}{2+\delta+2\delta^2}\right)\right]^{-1}. \tag{32}$$

Here $\delta = 1$ (or $\beta = \infty$) corresponds to a solid particle, $\delta = 0$, to a droplet.

2.3.3. *Thin circular disk.* For a thin circular disk the axis of which is arbitrarily oriented in the flow, the determining parameters are specified by the formulae

$$\begin{aligned} &Sh(1,0) = \frac{2}{\pi} \approx 0.637, \quad (\mathbf{F}\cdot\mathbf{i}) = F_{\perp}\cos^2\varepsilon + F_{\parallel}\sin^2\varepsilon,\\ &F_{\perp} = \frac{8}{3\pi} \approx 0.849, \quad F_{\parallel} = \frac{16}{9\pi} \approx 0.566 \quad (0 \leqslant \varepsilon \leqslant \pi/2). \end{aligned} \tag{33}$$

2.3.4. *Ellipsoid of revolution with the rotation axis directed along the flow.* Attention is now turned to a solid particle which represents an ellipsoid of revolution with the semi-axes a and b, where a is the equatorial radius positioned normally to the free stream, b is the polar radius directed along the flow. Having chosen the equatorial radius a as the characteristic length scale for the parameter $Sh(1,0)$ and the dimensionless resistance force, one obtains [21]

$$
\begin{aligned}
&Sh(1,0) = \begin{cases} (\chi^2+1)^{1/2}(\arctan \chi)^{-1}, & a \geqslant b, \\ (\chi^2-1)^{1/2}(\operatorname{arctanh} \chi)^{-1}, & a \leqslant b, \end{cases} \\
&(\mathbf{F}\cdot\mathbf{i}) = \tfrac{4}{3}(\chi^2-1)^{-1/2}\begin{cases} [\chi-(\chi^2-1)\arctan \chi]^{-1}, & a \geqslant b, \\ [(\chi^2+1)\operatorname{arctanh} \chi-\chi]^{-1}, & a \leqslant b, \end{cases} \\
&\chi = \left|\left(\frac{a}{b}\right)^2-1\right|^{-1/2}, \quad \operatorname{arctanh} \chi = \frac{1}{2}\ln\left(\frac{\chi+1}{\chi-1}\right).
\end{aligned}
\tag{34}
$$

2.3.5. *Particle composed of two contacting spheres.* Consider a solid 'dumb-bell' particle composed of two contacting spheres of radii a_1 and a_2. In this case for the mean Sherwood number $Sh(1,0)$ one obtains [22, 23]

$$
\begin{aligned}
Sh(1,0) &= -\frac{a_2}{a_1+a_2}\left[\psi\left(\frac{a_1}{a_1+a_2}\right)+\psi\left(\frac{a_2}{a_1+a_2}\right)+2\gamma\right], \\
\psi(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)\left(\lim_{x\to 0} x\psi(x) = -1, \quad \psi\left(\frac{1}{2}\right) = -\gamma-2\ln 2, \quad \psi(1) = -\gamma\right),
\end{aligned}
\tag{35}
$$

where the radius of the first sphere, a_1, is chosen for the characteristic length scale.

Some formulae to calculate the resistance force $(\mathbf{F}\cdot\mathbf{i})$ for such a particle having arbitrary orientation are given elsewhere [21].

Consider now a specific case of the particle composed of two contacting spheres of equal radii $a_1 = a_2 = a$. For the particle, the axis (passing through the centres of spheres) of which is arbitrarily oriented in space and which is at an angle ε to the direction of the free stream, one obtains [21–23]

$$
\begin{aligned}
&Sh(1,0) = 2\ln 2 \approx 1.386, \quad (\mathbf{F}\cdot\mathbf{i}) = F_\perp \cos^2\varepsilon + F_\parallel \sin^2\varepsilon, \\
&F_\perp \approx 0.707, \quad F_\parallel = 0.645 \quad (0 \leqslant \varepsilon \leqslant \pi/2).
\end{aligned}
\tag{36}
$$

2.4. *Unsteady-state mass exchange of a particle in a translational flow*

Consider now the unsteady-state mass exchange of a particle in a stationary translational flow, equation (9), in the case of a sudden occurrence of a reaction on its surface. The appropriate boundary-value problem is described by the equation and initial condition

$$
\frac{\partial c}{\partial t}+(\mathbf{v}\cdot\operatorname{grad} c) = Pe^{-1}\operatorname{div}(D\operatorname{grad} c); \quad t = 0, \quad c = 0,
\tag{37}
$$

subject to the boundary conditions (2).

This problem has a unique solution which at $t \to \infty$ becomes stationary and corresponding to equations (1) and (2).

By replacing equation (5) one passes in equation (37) and boundary conditions (2) from the concentration c to a new variable Φ.

The analysis is started with the study of the outer region Ω_∞, where the solution is sought in the form of expansion (11) with account for the fact that, besides everything else, the functions $\Phi^{(i)}$ depend also on time t. The zero term of the expansion, $\Phi^{(0)}$, just as in the stationary case, is equal to zero, while for the first term of the outer expansion the following linear equation is obtained

$$
\begin{aligned}
&Pe^{-1}\frac{\partial\Phi^{(1)}}{\partial t}+(\mathbf{i}\cdot\operatorname{grad}_z \Phi^{(1)}) = \Delta_z\Phi^{(1)}; \quad t = 0, \quad \Phi^{(1)} = 0; \\
&z \to \infty, \quad \Phi^{(1)} \to 0.
\end{aligned}
\tag{38}
$$

At $t \leqslant O(Pe^{-1})$, the second and third terms in equation (38) can be neglected, which, by virtue of the initial condition, yields $\Phi^{(1)} = 0$; at $t > O(Pe^{-1})$, the first term in equation (38) can be neglected which corresponds to the stationary case, equation (17). In the general case when $0 \leqslant t \leqslant \infty$ ($t = O(Pe^{-1})$), all the terms of equation (38) should be taken into account.

Applying to equation (38) the Laplace–Carlson transformation with respect to the variable t

$$
\Phi_*^{(1)} = \int_0^\infty e^{-pt}\,\Phi^{(1)}\,\mathrm{d}t,
$$

yields

$$
p\,Pe^{-1}\,\Phi_*^{(1)}+(\mathbf{i}\cdot\operatorname{grad}_z \Phi_*^{(1)}) = \Delta_z\Phi_*^{(1)}; \quad z \to \infty, \quad \Phi_*^{(1)} \to 0.
\tag{39}
$$

The solution of equation (39), which at $t \to \infty$ $(p \to 0)$ becomes stationary, equation (18), has the form [7]

$$\Phi_*^{(1)} = \frac{q}{z} \exp\left\{\frac{z}{2}\left[\eta - \left(1 + \frac{4p}{Pe}\right)^{1/2}\right]\right\}. \tag{40}$$

In much the same way as it was done earlier for the stationary case (see also refs. [6, 11]), one can show that the second term of the outer expansion (11) is determined by the solution of a nonhomogeneous equation, the differential (homogeneous) part of which coincides with equation (38). At $z \to 0$ $(t = \text{const.})$, the specific, damping at infinity, solution of this nonhomogeneous equation contains the same logarithmic term as that used in the stationary case.

Further it will be everywhere supposed that

$$t \geqslant O(Pe^{-1}) \quad (\text{i.e. } |p| \leqslant O(Pe)). \tag{41}$$

Taking into account what was said previously and expanding equation (40) into a series at $z \to 0$ with a subsequent transition from the transforms to the inverse transforms, one obtains that, subject to condition (41), an approximation for the asymptote to the function Φ, equation (10), in the inner region Ω_1 at $r \to \infty$ [the condition of matching with the outer expansion (11)] can be presented as

$$\begin{gathered} \Phi = \beta(\tau, Pe) + \tfrac{1}{2}Pe\, q\eta g(\tau) + O(Pe^2) + O(r^{-1}), \\ \beta(\tau, Pe) = -\tfrac{1}{2}q[Pe\, g(\tau) + Pe^2 \ln Pe(\mathbf{F}\cdot\mathbf{i})], \end{gathered} \tag{42}$$

where the function $g = g(\tau)$ is determined by the formula

$$g(\tau) = \frac{2}{\sqrt{(\pi\tau)}} e^{-\tau} + \operatorname{erf}\sqrt{\tau}, \quad \tau = \tfrac{1}{4}Pe\, t. \tag{43}$$

It is not difficult to show that on compliance of condition (41) the terms of the inner expansion $\Phi_i(t, r, \theta, \phi)$ [equation (10)] are determined by the same equations (12)–(14) as those used in the stationary case. Owing to this and by virtue of the boundary conditions at infinity [i.e. the conditions of matching with the outer expansions (42) and (43)], the terms of the inner expansion depend on time only parametrically; in this case the zero term of expansion Φ_0 does not depend on time and is prescribed by formulae (15) and (16).

Omitting the intermediate calculations, which, taking into account equations (42) and (43), are carried out in exactly the same manner as it is done in the stationary case, one again arrives at formula (28), where the mean Sherwood number, corresponding to the linear $(D = 1)$ nonstationary problem, has the form

$$Sh(1, Pe) = Sh(1, 0) + \tfrac{1}{2}Sh^2(1, 0)\{Pe\, g(\tau) + Pe^2 \ln Pe(\mathbf{F}\cdot\mathbf{i})\} + O(Pe^2). \tag{44}$$

Here, as before, $Sh(1, 0)$ corresponds to the steady-state mass exchange of the particle in a stationary medium at the constant diffusion coefficient $D = 1$. At $t \to \infty$ $(g(\tau) \to 1)$, expressions (28) and (44) pass into the solution of the stationary problem, equations (28) and (29).

It should be noted that formula (44), corresponding to the linear case $D = 1$, extends the results of ref. [6], obtained for a solid sphere [which corresponds to the values $Sh(1, 0) = 1$ and $(\mathbf{F}\cdot\mathbf{i}) = 1$ in equation (44)], to the case of an arbitrarily shaped solid or liquid particle.

2.5. *Certain correlations: convective heat and mass transfer of a particle in a liquid*

Similarly, one can show that, subject to condition (41), the formulae

$$Sh(\sigma_{ij}, Pe) = J(f)Sh(\omega_{ij}, Pe), \tag{45}$$

$$Sh(\omega_{ij}, Pe) = Sh(\omega_{ij}, 0) + \tfrac{1}{2}Sh^2(\omega_{ij}, 0)\{Pe\, g(\tau) + Pe^2 \ln Pe(\mathbf{F}\cdot\mathbf{i})\} + O(Pe^2), \tag{46}$$

determine the asymptotic behaviour of the mean Sherwood number for the following more general equation

$$\begin{gathered} \frac{\partial c}{\partial t} + (\mathbf{v}\cdot\operatorname{grad} c) = Pe^{-1}\frac{\partial}{\partial x_i}\sigma_{ij}(\mathbf{r}, c)\frac{\partial c}{\partial x_j}; \quad \sigma_{ij}(\mathbf{r}, c) = \omega_{ij}(\mathbf{r})f(c), \\ \omega_{ij} = \omega_{ji}, \quad \lim_{r\to\infty}\omega_{ij} = \delta_{ij}, \quad \sigma_{ij}x_ix_j \geqslant \sigma_0 x_i x_i, \quad \sigma_0 > 0, \end{gathered} \tag{47}$$

with the boundary conditions (2) and initial condition $t = 0$, $c = 0$; here and hereafter the summation over repeated indices i and j is employed. The mean Sherwood numbers figuring in equations (45) and (46) are determined in this case by

$$Sh(\sigma_{ij}, Pe) = -\frac{1}{4\pi}\int_S \sigma_{ij}\frac{\partial c}{\partial x_j}\cos(nx_i)\,\mathrm{d}S,$$

which, at $\omega_{ij}(\mathbf{r}) = \delta_{ij}, f(c) = D(c)$, coincides with equation (8).

A proof for this statement is provided by introduction of a new variable $\Phi = \Phi(f, c)$ by formula (5) with account for the fact that the first term of the outer expansion, $\Phi^{(1)}$ (as $\Phi^{(0)} = 0$ earlier), is also described by equation (38). Further it is shown in a similar way that the representation of equation (42) is valid on the inner boundary of the outer region Ω_∞. By virtue of equation (41), the terms of the inner expansion (10) are described by the corresponding stationary equations, the solution of which, due to the boundary conditions at infinity, equation (42), depend parametrically on time t. The zero term of the inner expansion, owing to the properties of the function σ_{ij}, can also be presented in the form of equations (15) and (16). By multiplying the corresponding equations for the terms of the inner expansion by Φ_0, $Pe\,\Phi_0$, $Pe^2 \ln Pe\,\Phi_0$ and making the addition, one arrives at the equality similar to equation (22), namely

$$\Phi_0 \frac{\partial}{\partial x_i} \omega_{ij} \frac{\partial \Phi}{\partial x_j} = Pe\,\Phi_0(\mathbf{v}\cdot \operatorname{grad} c(\Phi_0)).$$

Now integrate this expression over the control liquid volume V bounded by the particle surface S and the surface Σ_r of a sphere with a large enough radius. In integration one takes into account the second identity in system (23) and the equality

$$\Phi_0 \frac{\partial}{\partial x_i} \omega_{ij} \frac{\partial \Phi}{\partial x_j} = \frac{\partial}{\partial x_i} \Phi_0 \omega_{ij} \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i} \Phi \omega_{ij} \frac{\partial \Phi_0}{\partial x_j} + \Phi \frac{\partial}{\partial x_i} \omega_{ij} \frac{\partial \Phi_0}{\partial x_j},$$

in which the last term is equal to zero due to the equation for the zero term of the inner expansion. The integrals are evaluated with the aid of the third Green's formula which allows the transformation from volumetric to surface integrals, which ultimately leads to the required equations (45) and (46).

For a practically important example of the use of equations (45) and (46), corresponding to equation (47), consider the case of simultaneous heat and mass transfer of a particle in an incompressible liquid at small Péclet numbers. It is assumed that all the determining parameters of the system are temperature dependent. In ordinary liquids the thermal Péclet number, Pe_T, is much less than the diffusional Péclet number, Pe_c

$$Pe_T \ll Pe_c = Pe. \tag{48}$$

Therefore, at $Pe < O(1)$, in order to determine the integral inflow of substance to the particle, one may discard the convective term in the first approximation to the heat conduction equation and restrict the study to the following dimensionless set of equations and boundary conditions

$$\operatorname{div}(\lambda \operatorname{grad} T) = 0; \quad r = r_s, \quad T = 1; \quad r \to \infty, \quad T \to 0, \tag{49}$$

$$\begin{gathered}(\mathbf{v}\cdot\operatorname{grad} c) = Pe_c^{-1} \operatorname{div}(D \operatorname{grad} c); \quad r = r_s, \quad c = 1; \quad r \to \infty, \quad c \to 0, \\ T = \frac{T_\infty - T_*}{T_\infty - T_s}, \quad c = \frac{c_\infty - c_*}{c_\infty - c_s}, \quad Pe_c = \frac{aU}{D_*(T_\infty)}, \\ \lambda = \lambda(T) = \lambda_*(T_*)/\lambda_*(T_\infty), \quad D = D(T) = D_*(T_*)/D_*(T_\infty).\end{gathered} \tag{50}$$

Here λ_* is the thermal conductivity of liquid; $T_s \neq T_\infty$, $c_s \neq c_\infty$.

Equations (49) and (50) can be solved in succession. First, by solving the problem, equation (49), the temperature distribution outside the particle is constructed. Then, the substitution of the known function $T = T(\mathbf{r})$, which depends on the coordinates, into the expression for $D = D(T)$ determines the linear problem, equation (50), for concentration distribution in the flow.

The analysis of equation (49) shows that the mean Nusselt number, which determines the heat inflow to the particle surface, is of the form

$$Nu(\lambda, 0) = J(\lambda) Sh(1, 0) \quad \left(Nu(\lambda, Pe_T) = -\frac{1}{4\pi} \int_S \lambda(T) \frac{\partial T}{\partial n} \, \mathrm{d}S \right). \tag{51}$$

The equation for the concentration distribution, equation (50), is a specific case of the general stationary equation (47) at

$$\sigma_{ij}(\mathbf{r}, c) = D(T)\delta_{ij}, \quad T = T(\mathbf{r}) \quad (\omega_{ij} = D(T(\mathbf{r}))\delta_{ij}, \quad f(c) = 1). \tag{52}$$

Therefore, equation (46) remains valid (at $g \equiv 1$) for the mean Sherwood number, with the parameter $Sh(\omega_{ij}, 0) \equiv Sh(D, 0)$ corresponding to the mass transfer of a particle in a heated stationary liquid

$$\operatorname{div}(D(T) \operatorname{grad} c_0) = 0; \quad r = r_s, \quad c_0 = 1; \quad r \to \infty, \quad c_0 \to 0. \tag{53}$$

The solution of problem (53) is sought in the form

$$c_0 = y(T) y^{-1}(1), \quad y(0) = 0, \tag{54}$$

where the function y is determined by substituting this equation into equation (53) with a subsequent comparison with equation (49) for the temperature distribution. As a result one obtains

$$y(T) = \int_0^T \frac{\lambda(\xi)}{D(\xi)}\,\mathrm{d}\xi = \Phi\left(\frac{\lambda}{D}, T\right). \tag{55}$$

The mean Sherwood number, corresponding to the mass exchange of a particle in a stationary liquid, equation (53), can be presented, by virtue of equations (54) and (55), in the form

$$Sh(D,0) = J^{-1}(\lambda/D)Nu(\lambda,0).$$

This formula, with account for equations (46) and (51), leads to the following final expression for the mean Sherwood number

$$\begin{gathered} Sh(D,Pe) = Sh(D,0)+\tfrac{1}{2}Sh^2(D,0)\{Pe_c+Pe_c^2 \ln Pe_c(\mathbf{F}\cdot\mathbf{i})\}+O(Pe_c^2)+O(Pe_T), \\ Sh(D,0) = J(\lambda)J^{-1}(\lambda/D)Sh(1,0). \end{gathered} \tag{56}$$

In the linear case, when $D = 1$, equation (56) passes over into equation (29).

2.6. *Mass exchange of a particle in an arbitrary shear flow*

In the case of a particle streamlined by a uniform shear flow of incompressible liquid of an arbitrary order n, the velocity field at infinity is determined by

$$\begin{gathered} r \to \infty, \quad \mathbf{v} = \mathbf{Q}^{(n)}(\mathbf{r})+O(r^{-1}), \quad \operatorname{div} \mathbf{Q}^{(n)} = 0, \\ \mathbf{Q}^{(n)}(\kappa\mathbf{r}) = \kappa^n\mathbf{Q}^{(n)}(\mathbf{r}), \quad \mathbf{Q}^{(n)} = (Q_1^{(n)}(r),\ Q_2^{(n)}(r),\ Q_3^{(n)}(r)). \end{gathered} \tag{57}$$

In the case of a simple linear shear flow ($n = 1$), one obtains

$$\begin{gathered} Q_i^{(1)} = G_{ij}x_j, \quad G_{ii} = 0 \quad (i,j = 1,2,3), \\ (G_{ij} = G_{ij}^*G^{-1}, \quad G = \max_{i,j}|G_{ij}^*|, \quad U = aG). \end{gathered} \tag{58}$$

Next, investigate the nonlinear boundary-value problem [equations (6), (7) and (57)] by the method of matched asymptotic expansions in small Péclet number. In this case the entire flow region is divided into two subregions:

the inner $\Omega_1^{(v)} = \{r_s \leqslant r \leqslant O(Pe^{-v})\}$,

and

the outer $\Omega_\infty^{(v)} = \{O(Pe^{-v}) \leqslant r\}$,

where the parameter v and the 'contracted' coordinate are determined by the expressions

$$v = (n+1)^{-1}, \quad z_v + Pe^v r. \tag{59}$$

The zero term of the inner expansion, Φ_0, just as before, is prescribed by equation (15), while the zero term of the outer expansion is equal to zero.

Consider first the linear problem [equations (1) and (2)] at $\sigma = 1$ ($n \neq 0$). The solution in each of the above subregions is sought in the form of the expansion in integral powers of the parameter

$$c = \sum_{i=0}^{n+2} Pe^{vi}\,c_i+O(Pe^{1+2v}), \quad c_i = c_i(r,\theta,\phi) \quad (\text{in } \Omega_1^{(v)}), \tag{60}$$

$$c = \sum_{i=0}^{n+2} Pe^{vi}\,c^{(i)}+O(Pe^{1+2v}), \quad c^{(i)}(z_v,\theta,\phi), \quad c^{(0)} = 0 \quad (\text{in } \Omega_\infty^{(v)}). \tag{61}$$

The substitution of the outer expansion into equation (1) shows, taking into account equations (57) and (59), that all of the written out terms of the outer expansion (61) satisfy one and the same equation with the boundary condition of damping at infinity $z_v \to \infty, c^{(i)} \to 0$. Due to this, the solution in the outer region can be replaced more conveniently by

$$c = Pe^v\,\Psi(Pe)\{\zeta+O(Pe^{1+v})\} \quad (\Psi(0) = q), \tag{62}$$

where $\Psi(Pe)$ is the unknown function of the parameter Pe [which is the $(n+1)$th degree polynomial of the quantity Pe^v], which will be determined in the course of the problem solution, while ζ is the fundamental solution of the equation

$$\Delta_{z_v}\zeta = (\mathbf{Q}^{(n)}\cdot\operatorname{grad}_{z_v}\zeta), \tag{63}$$

which has the following limiting properties

$$z_\nu \to 0, \quad \zeta \to z_\nu^{-1} - \alpha + \gamma(\mathbf{z}_\nu) + o(1), \quad \alpha = \alpha(\mathbf{Q}^{(n)}),$$
$$\int_{\Sigma_r} \gamma(\mathbf{z}_\nu)\, \mathrm{d}\Sigma_r = 0 \quad (\gamma(\mathbf{z}_\nu) = O(1)). \tag{64}$$

Here α is the numerical factor, the value of which depends on the liquid velocity field at infinity [equation (57)] and which should be determined in each specific case.

For the translational flow past a particle [equation (9)] ($n = 0, Q_i^{(0)} = \delta_{10}$), equation (18) yields $\alpha = 1/2, \gamma = \eta/2$. The general expression to determine the parameter $\alpha = \alpha(G_{ij})$ in the case of an arbitrary linear ($n = 1$) shear flow [equation (58)] has been obtained in ref. [8] (at $n = 1$ the equality $\gamma = 0$ is always valid). In particular, in the axisymmetric case ($G_{11} = G_{22} = 1/2, G_{33} = -1; G_{ij} = 0$ at $i \neq j$), one obtains $\alpha = 0.399$ [8], while in the case of a simple shear (one off-diagonal element of the matrix G_{ij} is equal to unity, while the remainder is equal to zero), $\alpha = 0.258$ [7].

By substituting equation (60) into equations (1) and (2), one obtains the equations and boundary conditions for the terms of the inner expansion

$$\Delta c_i = 0; \quad i = 0, 1, \ldots, n; \quad \Delta c_{n+1} = (\mathbf{v} \cdot \operatorname{grad} c_0), \quad \Delta c_{n+2} = (\mathbf{v} \cdot \operatorname{grad} c_1),$$
$$r = r_s, \quad c_0 = 1, \quad c_i = 0 \quad (i = 1, 2, \ldots, n+2). \tag{65}$$

Multiply the ith equation by $Pe^{i\nu}$ and sum the first n equations of system (65). This yields $\Delta c = 0$, where c is determined by the sum of equation (60) with the summation limit $n+2$ replaced by n. Further, the integration of $\Delta c = 0$ and $c_0 \Delta c = 0$, with account for equations (62) and (64), over the control liquid volume V and transformation to the surface integrals yield two equations to determine the mean Sherwood number [equation (8)] and the unknown function Ψ

$$Sh(1, Pe) = \Psi(Pe)(1 + O(Pe)),$$
$$Sh(1, Pe) = Sh(1, 0) + Sh(1, 0)\Psi(Pe)(\alpha + O(Pe)). \tag{66}$$

By solving these equations, one arrives at the following expression for the mean Sherwood number

$$\frac{Sh(1, Pe)}{Sh(1, 0)} = \sum_{i=0}^{n} \alpha^i\, Sh^i(1, 0) Pe^{\nu i} + O(Pe) \approx [1 - \alpha\, Sh(1, 0) Pe^\nu]^{-1}. \tag{67}$$

For a spherical particle ($Sh(1, 0) = 1$) the accuracy of equation (67) increases up to $O(Pe^{1+\nu})$; in this case the upper summation limit n should be replaced by $n+1$. This statement is proved by the summation of the $(n+1)$ first equations of system (65) multiplied by $c_0\, Pe^{i\nu}$, respectively, with the subsequent integration over the control liquid volume V with account for the fact that the zero term of the inner expansion has, in this case, the form $c_0 = r^{-1}$. As a result, one arrives at expression (24), where $\Phi = c, \Phi_0 = c_0, E = c_0^2/2$ and the first five integrals are calculated in the same way as was done earlier. When evaluating the integral I_6 over the sphere surface Σ_r it is taken into account that the function $E = c_0^2/2 = r^{-2}/2$ can be taken outside the integral sign. Moreover, the integral over Σ of $(\mathbf{v} \cdot \mathbf{n})$ is equal, by virtue of equation (3), to the integral of the same quantity over the sphere surface S, which, due to equation (4), leads to the equality $I_6 = 0$. Similarly one can show the vanishing of the volumetric integral over the volume V of the sum on the RHS of the first $(n+1)$ equations of system (65) multiplied respectively by $Pe^{i\nu}$. What has just been said offers the possibility to obtain equation (66) for the mean Sherwood number, where $O(Pe)$ is replaced by $O(Pe^{1+\nu})$. Moreover, if in equation (64) $\gamma(\mathbf{z}_\nu) = 0$, then the accuracy of equation (67) increases up to $O(Pe^{1+2\nu})$ and the upper summation limit in equation (67) should be replaced by $n+2$.

In the general case, for the analysis of the nonlinear equations (6), (7) and (57), one introduces, besides equation (5), a new additional variable

$$\Lambda = \Lambda(c) = J^{-1}(D)\Phi(D, c), \tag{68}$$

which differs from the variable $\Phi = \Phi(D, c)$ only by normalization.

At $Pe \to 0$, similarly to the linear case, the solution in the inner, $\Omega_1^{(\nu)}$, and outer, $\Omega_\infty^{(\nu)}$, regions is sought in the form ($n \geqslant 1$)

$$\Lambda = \Lambda_0 + Pe^\nu \Lambda_1 + O(Pe^{2\nu}), \quad \Lambda_i = \Lambda_i(r, \theta, \phi); \quad i = 0, 1 \quad (\text{in } \Omega_1^{(\nu)}), \tag{69}$$

$$\Lambda = \Lambda^{(0)} + Pe^\nu \Lambda^{(1)} + O(Pe^{2\nu}), \quad \Lambda^{(0)} = 0, \quad \Lambda^{(1)} = \Lambda^{(1)}(z_\nu, \theta, \phi) \quad (\text{in } \Omega_\infty^{(\nu)}). \tag{70}$$

The terms of the inner expansion (69) are determined by solving the Laplace equations with the boundary conditions on the particle surface

$$\Delta\Lambda_0 = 0, \quad \Delta\Lambda_1 = 0; \quad r = r_s, \quad \Lambda_0 = 1, \quad \Lambda_1 = 0, \tag{71}$$

while the first term of the outer expansion (70) satisfies the equation

$$\Delta_{z_\nu}\Lambda^{(1)} = (\mathbf{Q}^{(n)}\cdot \operatorname{grad}_{z_\nu} \Lambda^{(1)}); \quad z_\nu \to \infty, \quad \Lambda^{(1)} \to 0 \quad (z_\nu = Pe^\nu r). \tag{72}$$

The first two terms of the inner and outer expansions, equations (69)–(72), exactly satisfy the same equations ($\Psi(Pe) \to \Psi(0)$) and boundary conditions (62)–(65), as in the linear case $D = 1$, $\Lambda = c$. Therefore, with account for equations (5) and (68) one obtains that, in the case of an arbitrary shear flow past a particle of any shape, equation (57), the two-term expansion of the mean Sherwood number, equation (8), in Péclet number is also determined by equation (28), where the parameter $Sh(1, Pe)$ has the form

$$Sh(1, Pe) = Sh(1,0) + \alpha\, Pe^\nu\, Sh^2(1,0) + O(Pe^{2\nu}). \tag{73}$$

For an arbitrary linear shear flow [equation (58)] ($n = 1$, $\nu = 1/2$), equations (28) and (73) extend the results obtained by Batchelor [8] to a nonlinear case (the general expression to determine the magnitude of the parameter $\alpha = \alpha(G_{ij})$ is given in ref. [8]).

Equations (28) and (73) remain valid also for an arbitrary nonuniform flow, when the vector-function $\mathbf{Q}^{(n)}(\mathbf{r})$ is not homogeneous in $\mathbf{r}$ and has the property $\mathbf{Q}^{(n)}(\kappa\mathbf{r}) = \kappa^n \mathbf{Q}_n^{(n)}(\mathbf{r}) + \kappa^{n-1}\mathbf{Q}_{n-1}^{(n)}(\mathbf{r}) + \ldots$ In this case the parameter α, in equation (73), is defined by the approximation for the asymptote (64) to the fundamental solution of equation (63), where $\mathbf{Q}^{(n)}$ is replaced by $\mathbf{Q}_n^{(n)}$.

Equations (28) and (73) are also valid for the case of mass exchange of a particle of any shape freely suspended in an arbitrary linear ($n = 1$) shear flow, equation (58). Here, due to particle rotation, one should consider a nonstationary counterpart of equations (1) and (2). The introduction of the variable Λ by formulae (5) and (68) shows that in this case too the first two terms of the corresponding inner and outer expansions are described by the same equations and boundary conditions as in the linear case. Taking into account the fact that in the linear case equation (73) remains valid, as shown in ref. [9], ($\nu = 1/2$), one arrives at the result required.

In the case of simultaneous heat and mass exchange of a particle in a liquid, equations (49), (50) and (57), the mean Sherwood number per particle will be preassigned, accurate to $O(Pe^{2\nu})$, by

$$Sh(D, Pe) = Sh(D,0) + \alpha\, Pe^\nu\, Sh^2(D,0), \quad Sh(D,0) = J(\lambda)J^{-1}(\lambda/D)Sh(1,0),$$

which, at $D = 1$, passes into equation (73).

Remark. The respective thermal problems are considered similarly in the case of a particle (droplet) of an arbitrary shape streamlined by a translational, equation (9), or shear, equation (57), flow of an incompressible liquid, the thermal conductivity, λ, and specific heat, c_p, of which depend on the temperature T. The replacement of T by a new variable h following the equation $h = x(T)x^{-1}(1)$, $x = \int_0^T c_p(\xi)\,\mathrm{d}\xi$ reduces the thermal problem to the equation of convective diffusion, equation (1), with the boundary conditions (2) [after having replaced $c \to h$, $D \to (\lambda/c_p)_{T=T(h)}$]. From what has been said it follows, after elementary calculations, that equation (28) remains, after the replacements $Sh \to Nu$, $D \to \lambda$, valid also for thermal problems.

3. SIMULTANEOUS HEAT AND MASS TRANSFER TO A PARTICLE IN THE FLOW OF A COMPRESSIBLE GAS WITH THE DETERMINING PARAMETERS ARBITRARILY DEPENDENT ON TEMPERATURE

3.1. *Statement of the problem*

Consider a stationary heat and mass exchange between an arbitrarily shaped particle (droplet) and the translational flow of a compressible gas the determining parameters of which depend arbitrarily on temperature. It is assumed that concentration and temperature on the particle surface and far from it (at infinity) take on constant values. By neglecting the effects of heat and pressure diffusion and omitting the terms of the squared Mach number order, one can write down in dimensionless variables the corresponding boundary-value problems as

$$Pe_{\mathrm{T}}\, c_p \rho(\mathbf{v}\cdot \operatorname{grad} T) = \operatorname{div}(\lambda \operatorname{grad} T), \tag{74}$$

$$Pe_{\mathrm{c}}\, \rho(\mathbf{v}\cdot \operatorname{grad} u) = \operatorname{div}(\sigma \operatorname{grad} u), \tag{75}$$

$$r = r_{\mathrm{s}}, \quad T = 1; \quad r \to \infty, \quad T \to 0, \tag{76}$$

$$r = r_{\mathrm{s}}, \quad u = 1; \quad r \to \infty, \quad u \to 0, \tag{77}$$

$$T = \frac{T_\infty - T_*}{T_\infty - T_{\mathrm{s}}}, \quad u = \frac{u_\infty - u_*}{u_\infty - u_{\mathrm{s}}}, \quad \rho = \frac{\rho_*}{\rho_\infty}, \quad c_p = \frac{c_{p*}(T_*)}{c_{p*}(T_\infty)},$$

$$Pe_{\mathrm{T}} = Re_\infty\, Pr_\infty, \quad Pe_{\mathrm{c}} = Re_\infty\, Sc_\infty \quad (T_{\mathrm{s}} \neq T_\infty, \quad u_{\mathrm{s}} \neq u_\infty),$$

$$Re = aU\rho_*\mu_*^{-1}, \quad Pr = \mu_* c_{p*}\lambda_*^{-1}, \quad Sc = \mu_*(\rho_* D_*)^{-1},$$

$$\sigma \equiv \rho D = \sigma(T), \quad \lambda = \lambda(T), \quad c_p = c_p(T).$$

Here it is assumed that the parameters c_{p*}, λ_*, D_*, μ_* and the Schmidt number Sc depend on temperature T_* in the conventional manner, while the quantities $\mathbf{v}$, ρ, and T are determined by solving the full problem of the streamlining of the particle by a viscous heat conducting compressible gas flow. The subsequent analysis will require only the continuity equation

$$\operatorname{div}(\rho \mathbf{v}) = 0, \tag{78}$$

as well as the condition that the normal gas velocity component on the particle surface be zero [equation (4)]. (The equations of motion will not be required for the further analysis.)

It is usually assumed that the gas has a constant heat capacity and constant Prandtl, $Pr = Pr_\infty$, and Schmidt, $Sc = Sc_\infty$, numbers and that the gas viscosity depends conventionally on temperature. These assumptions correspond to the following values of the parameters entering into equations (74) and (75)

$$\lambda = \sigma = \mu \equiv \mu(T) \quad (\mu(T) = \mu_*(T_*)/\mu_*(T_\infty)). \tag{79}$$

Certain specific functions $\mu = \mu(T)$ will be considered in Section 3.5.

Further, the study of the problem, equations (74)–(77), will be based on the most general case, when the parameters λ, σ, and c_p are arbitrary functions of the dimensionless temperature T.

As before, to simplify the analysis, the temperature T is replaced by a new variable

$$\Phi = \Phi(\lambda, T) = \int_0^T \lambda(\xi)\,\mathrm{d}\xi. \tag{80}$$

The initial boundary-value problem, equations (74)–(77), with regard for the identities

$$c_p \rho(\mathbf{v} \cdot \operatorname{grad} T) \equiv \operatorname{div}(\rho \mathbf{v} h) - h \operatorname{div}(\rho \cdot \mathbf{v}), \quad h(T) = \int_0^T c_p(\xi)\,\mathrm{d}\xi,$$

$$\rho(\mathbf{v} \cdot \operatorname{grad} u) \equiv \operatorname{div}(\rho \mathbf{v} u) - u \operatorname{div}(\rho \mathbf{v}),$$

and the continuity equation (78) is reformulated in terms of the function Φ as

$$Pe_{\mathrm{T}} \operatorname{div}(\rho \mathbf{v} h) = \Delta\Phi; \quad r = r_{\mathrm{s}}, \quad \Phi = J(\lambda); \quad r \to \infty, \quad \Phi \to 0, \tag{81}$$

$$\begin{aligned} &Pe_{\mathrm{c}} \operatorname{div}(\rho \mathbf{v} u) = \operatorname{div}(\sigma \operatorname{grad} u); \quad r = r_{\mathrm{s}}, \quad u = 1; \quad r \to \infty, \quad u \to 0, \\ &h = h(\Phi) \equiv h(T(\Phi)), \quad \sigma = \sigma(\Phi) \equiv \sigma(T(\Phi)). \end{aligned} \tag{82}$$

Here, the function $T = T(\Phi)$ is determined by the function reversion, equation (80).

If it is assumed that the thermal and diffusional Péclet numbers are small and are of the same order (the latter property is typical of gases)

$$Re_\infty \to 0, \quad Pr_\infty = O(1), \quad Sc_\infty = O(1) \quad (Pe_{\mathrm{T}} = Re_\infty Pr_\infty, \; Pe_{\mathrm{c}} = Re_\infty Sc_\infty). \tag{83}$$

3.2. *The method of solution: auxiliary equations*

Investigate the problem, equations (74)–(77) [or equations (81) and (82)] by the method of matched asymptotic expansions in the small parameter Re_∞. As before, the solution is sought separately in the inner, $\Omega_1 = \{r_s \leqslant r \leqslant O(Re_\infty^{-1})\}$, and outer, $\Omega_\infty = \{O(Re_\infty^{-1}) \leqslant r\}$, regions of the flow in the following form

$$\begin{aligned} &\Phi = \Phi_0 + Pe_{\mathrm{T}} \Phi_1 + o(Re_\infty), \quad u = u_0 + Pe_{\mathrm{c}} u_1 + o(Re_\infty) \quad (\text{in } \Omega_1), \\ &\Phi_i = \Phi_i(r, \theta, \phi), \quad u_i = u_i(r, \theta, \phi); \quad i = 0, 1, \end{aligned} \tag{84}$$

$$\begin{aligned} &\Phi = Pe_{\mathrm{t}} \Phi^{(1)} + o(Re_\infty), \quad u = Pe_{\mathrm{c}} u^{(1)} + o(Re_\infty), \quad \Phi^{(0)} = u^{(0)} = 0 \quad (\text{in } \Omega_\infty), \\ &\Phi^{(1)} = \Phi^{(1)}(z_{\mathrm{T}}, \theta, \phi), \quad u^{(1)} = u^{(1)}(z_{\mathrm{c}}, \theta, \phi); \quad z_{\mathrm{T}} = Pe_{\mathrm{T}} r, \quad z_{\mathrm{c}} = Pe_{\mathrm{c}} r. \end{aligned} \tag{85}$$

Here and hereafter, the quantities are expanded, for convenience, in the parameters Pe_{T} and Pe_{c}.

Further analysis will require the solution of the simplest linear auxiliary problem for the conventional Laplace equation

$$\Delta c_0 = 0; \quad r = r_{\mathrm{s}}, \quad c_0 = 1; \quad r \to \infty, \quad c_0 \to 0, \tag{86}$$

which, owing to a proper selection of the origin for the radius-vector r, can be written in the form [2]

$$c_0 = Sh(1, 0) r^{-1} + O(r^{-3}). \tag{87}$$

By substituting equation (85) into equations (86) and (87) and assuming that $Pe_{\mathrm{T}} = Pe_{\mathrm{c}} = 0$, one obtains that the zero term of the inner expansion of the thermal part of the problem, Φ_0, is determined by equation (12) [after $J(D)$ is replaced by $J(\lambda)$], while the function u_0 is the solution of the following equation and boundary conditions

$$\operatorname{div}(\sigma(\Phi_0) \operatorname{grad} u_0) = 0; \quad r = r_{\mathrm{s}}, \quad u_0 = 1; \quad r \to \infty, \quad u_0 \to 0. \tag{88}$$

It is not difficult to show that the solution of these problems can be presented in terms of the auxiliary function c_0, equations (86) and (87), as

$$\Phi_0 = J(\lambda)c_0, \quad u_0 = y(\Phi_0)y^{-1}(J(\lambda)), \tag{89}$$

$$y(\Phi_0) = \int_0^{\Phi_0} \frac{\mathrm{d}\Phi}{\sigma(\Phi)} = \int_0^{T_0} \frac{\lambda(T)\,\mathrm{d}T}{\sigma(T)} = \Phi\left(\frac{\lambda}{\sigma}, T_0\right).$$

Here, the explicit form of the function y was determined by substituting the expression for u_0 into equation (88) and then making a comparison with the equation for Φ_0, equation (12). So that it be more readily evident, a corresponding expression for u_0 is also given in equation (89) through the zero term of the inner expansion of temperature T_0. This term is prescribed by reversing the expression $\Phi_0 = \Phi(\lambda, T_0)$.

The mean Sherwood and Nusselt numbers, determined by the zero terms of the inner expansions (88) and (89), have the form

$$Nu(\lambda, 0) = J(\lambda)Sh(1,0), \quad Sh(\sigma, 0) = J(\lambda)J^{-1}\left(\frac{\lambda}{\sigma}\right)Sh(1,0). \tag{90}$$

The second equation has been derived with account for the equality

$$y(J(\lambda)) = J(\lambda/\sigma).$$

Equations for the first terms of the outer expansion (85) can be obtained on the basis of the following limiting properties of the functions that determine the problem (81) and (82)

$$\rho \to 1, \quad \lambda \to 1, \quad \sigma \to 1, \quad T \to \Phi, \quad h \to \Phi, \quad \mathbf{v} \to \mathbf{i} \quad \text{at } r \to \infty. \tag{91}$$

The equations for $\Phi^{(1)}$ and $u^{(1)}$, accurate to the replacement $z \to z_{\mathrm{T}}$, $\Phi^{(1)} \to \Phi^{(1)}$ and $z \to z_{\mathrm{c}}$, $\Phi^{(1)} \to u^{(1)}$, coincide with equation (17). Therefore, the solutions of these equations, satisfying the condition of matching with the zero terms of the inner expansion (88) and (89), can be presented in the form

$$\Phi^{(1)} = Nu(\lambda, 0)z_{\mathrm{T}}^{-1} \exp\left[\tfrac{1}{2}z_{\mathrm{T}}(\eta - 1)\right], \quad u^{(1)} = Sh(\sigma, 0)z_{\mathrm{c}}^{-1} \exp\left[\tfrac{1}{2}z_{\mathrm{c}}(\eta - 1)\right].$$

The second terms of the expansion of these expressions into the series in small quantities z_{T} and z_{c}, by virtue of equation (85) and the matching condition, determine the following boundary conditions at infinity for the first terms of the inner asymptotic expansion (84)

$$r \to \infty, \quad \Phi_1 \to \tfrac{1}{2}Nu(\lambda, 0)(\eta - 1), \quad u_1 \to \tfrac{1}{2}Sh(\sigma, 0)(\eta - 1). \tag{92}$$

It is not difficult to show that the equations and boundary conditions, which correctly describe the first two terms of the inner expansion (84), are

$$\begin{gathered} Pe_{\mathrm{T}} \operatorname{div}(\rho\mathbf{v}h)_0 = \Delta\Phi, \quad \Phi = \Phi_0 + Pe_{\mathrm{T}}\Phi_1, \\ r = r_{\mathrm{s}}, \quad \Phi = J(\lambda); \quad r \to \infty, \quad \Phi \to \tfrac{1}{2}Pe_{\mathrm{T}}\, Nu(\lambda, 0)(\eta - 1), \end{gathered} \tag{93}$$

$$\begin{gathered} Pe_{\mathrm{c}} \operatorname{div}(\rho\mathbf{v}u)_0 = \operatorname{div}(\sigma(\Phi)\operatorname{grad} u), \quad u = u_0 + Pe_{\mathrm{c}}u_1, \\ r = r_{\mathrm{s}}, \quad u = 1; \quad r \to \infty, \quad u \to \tfrac{1}{2}Pe_{\mathrm{c}}\, Sh(\sigma, 0)(\eta - 1). \end{gathered} \tag{94}$$

The boundary-value problems are written down accurate to $o(Re_\infty)$, which can be verified by directly comparing the equations and boundary conditions (81), (82) and (93), (94) taking account of equations (84), (85), (12), (88) and (92); the subscript '0' on the LHS of equations (93) and (94) corresponds to the quantities containing the zero terms of the inner expansion, Φ_0 and u_0.

The solution of the inhomogeneous equations (93) and (94) is sought directly for the quantities Φ and u.

The solution of the thermal part of the problem (93) is sought in the form of the sum

$$\Phi = \bar{\Phi} + \delta\Phi, \tag{95}$$

where the terms satisfy the following equations and boundary conditions

$$\Delta\bar{\Phi} = 0; \quad r = r_{\mathrm{s}}, \quad \bar{\Phi} = J(\lambda); \quad r \to \infty, \quad \bar{\Phi} \to -\tfrac{1}{2}Pe_{\mathrm{T}}\, Nu(\lambda, 0), \tag{96}$$

$$\Delta\delta\Phi = Pe_{\mathrm{T}} \operatorname{div}(\rho\mathbf{v}h)_0; \quad r = r_{\mathrm{s}}, \quad \delta\Phi = 0; \quad r \to \infty, \quad \delta\Phi \to \tfrac{1}{2}Pe_{\mathrm{T}}\, Nu(\lambda, 0)\eta. \tag{97}$$

The distribution of the relative concentration is sought in the form

$$u = \bar{u} + \delta u, \tag{98}$$

where the terms are the solutions of the following boundary-value problems

$$\operatorname{div}(\delta(\bar{\Phi})\operatorname{grad} \bar{u}) = 0; \quad r = r_{\mathrm{s}}, \quad \bar{u} = 1; \quad r \to \infty, \quad \bar{u} \to -\tfrac{1}{2}Pe_{\mathrm{c}}\, Sh(\sigma, 0), \tag{99}$$

$$\begin{aligned} &\operatorname{div}(\sigma(\Phi)\operatorname{grad}\delta u) = -\operatorname{div}\{[\sigma(\Phi)-\sigma(\bar\Phi)]\cdot\operatorname{grad}\bar u\} + Pe_{\mathrm{c}}\operatorname{div}(\rho\mathbf{v}u)_0;\\ &r = r_{\mathrm{s}},\quad \delta u = 0;\quad r\to\infty,\quad \delta u \to \tfrac{1}{2}Pe_{\mathrm{c}}\,Sh(\sigma,0)\eta. \end{aligned} \tag{100}$$

Equations (96), (97), (99) and (100) yield that $\delta\Phi = O(Re_\infty)$, $\delta u = O(Re_\infty)$.

3.3. *The solution of the equations for the first terms: mean Sherwood and Nusselt numbers*

A direct check can easily show that the solution of problem (96) can be represented in terms of the function c_0, equations (86) and (87), as

$$\bar\Phi = [J(\lambda)+\tfrac{1}{2}Pe_{\mathrm{T}}\,Nu(\lambda,0)]c_0 - \tfrac{1}{2}Pe_{\mathrm{T}}\,Nu(\lambda,0). \tag{101}$$

The solution of problem (99) is sought in the form

$$\bar u = Ay(\bar\Phi)+B,\quad y(\bar\Phi) = \int_0^{\bar\Phi}\frac{\mathrm{d}\Phi}{\sigma(\Phi)}. \tag{102}$$

Due to the harmonicity of the function $\bar\Phi$, equation (96), equation (102) is the solution of equation (99) at any values of A and B. The explicit form of the constants A and B is determined by solving the linear algebraic system

$$1 = Ay(J(\lambda))+B;\quad -\tfrac{1}{2}Pe_{\mathrm{c}}\,Sh(\sigma,0) = Ay(-\tfrac{1}{2}Pe_{\mathrm{T}}\,Nu(\lambda,0))+B, \tag{103}$$

which follows from the boundary conditions on the particle surface and at infinity for the functions $\bar\Phi$ and $\bar u$ [equations (96) and (99)]. Restricting ourselves in the second of equations (103) to the principal term of expansion of the function y at $Pe_{\mathrm{T}}\to 0$ [the remaining terms need not be taken into account, since the initial system, equations (93) and (94) has only the accuracy $o(Re_\infty)$] and solving this system taking into account equation (90), one obtains the following expressions for the coefficients

$$\begin{aligned} A &= J^{-1}\left(\frac{\lambda}{\sigma}\right)\left[1+\frac{1}{2}(Pe_{\mathrm{c}}-Pe_{\mathrm{T}})J(\lambda)J^{-1}\left(\frac{\lambda}{\sigma}\right)Sh(1,0)\right]+o(Re_\infty),\\ B &= -\frac{1}{2}(Pe_{\mathrm{c}}-Pe_{\mathrm{T}})J(\lambda)J^{-1}\left(\frac{\lambda}{\sigma}\right)Sh(1,0)+o(Re_\infty). \end{aligned} \tag{104}$$

In a nonperturbed flow far from the particle at $r\to\infty$ the structure of the asymptotic expansion of the functions $\delta\Phi$ and δu should be the same for a particle of any shape. It will be shown in Section 3.4 that for a sphere the approximation for the asymptote to the solution of equations (97) and (100) at $r\to\infty$ does not involve the 'source' terms proportional to r^{-1} and, consequently, this very property is typical of the corresponding solutions for particles of any shape.

Now, integrate equations (97) and (100) over the control gas volume V with the subsequent integration over the surfaces S and Σ_r. The integrals over the particle surface S corresponding to the RHS of equations (97) and (100) vanish due to the nonpenetrability condition [equation (4)] and boundary conditions at $r = r_{\mathrm{s}}$ for the functions Φ and $\bar\Phi$ [equations (93) and (96)]. The integrals over the sphere surface Σ_r, corresponding to the RHS of equations (97) and (100), also vanish as $r\to\infty$ due to equations (89) and (91). The integrals over $\Sigma_{r=\infty}$ of the LHS of the equations vanish owing to the absence of 'source' terms in the expansions of $\delta\Phi$ and δu at $r\to\infty$. This, in turn, means that the remaining integrals over the particle surface S also vanish, i.e. the second terms, $\delta\Phi$ and δu [equations (95) and (98)], do not contribute to the mean Nusselt and Sherwood numbers, so that $\delta Nu = \delta Sh = 0$.

Taking into account what was said above and using equations (101), (102) and (104) for the mean Nusselt and Sherwood numbers, one obtains, accurate to $o(Re_\infty)$

$$\begin{aligned} Nu(\lambda,Pe_{\mathrm{T}}) &= J(\lambda)Sh(1,0)[1+\tfrac{1}{2}Pe_{\mathrm{T}}\,Sh(1,0)],\\ Sh(\sigma,Pe) &= Sh(\sigma,0)[1+\tfrac{1}{2}Pe_{\mathrm{T}}\,Sh(1,0)+\tfrac{1}{2}(Pe_{\mathrm{c}}-Pe_{\mathrm{T}})Sh(\sigma,0)]\\ &= J^{-1}\left(\frac{\lambda}{\sigma}\right)Nu(\lambda,Pe_{\mathrm{T}})[1+\tfrac{1}{2}(Pe_{\mathrm{c}}-Pe_{\mathrm{T}})Sh(\sigma,0)],\\ Sh(\sigma,0) &= J(\lambda)J^{-1}(\lambda/\sigma)Sh(1,0). \end{aligned} \tag{105}$$

Similarly one can find the corresponding expressions for the shear flow [equation (57)]. In a general case of an arbitrary flow past a particle, the equations for the mean Nusselt and Sherwood numbers can be written, accurate to $o(Re_\infty^\nu)$, in the form

$$\begin{aligned} Nu(\lambda,Re_\infty) &= J(\lambda)Sh(1,0)[1+\alpha\,Re_\infty^\nu\,Pr_\infty^\nu\,Sh(1,0)],\\ Sh(\sigma,Re_\infty) &= Sh(\sigma,0)\{1+\alpha\,Re_\infty^\nu[Pr_\infty^\nu\,Sh(1,0)+(Sc_\infty^\nu-Pr_\infty^\nu)Sh(\sigma,0)]\}, \end{aligned} \tag{106}$$

where the values $\alpha = 1/2$, $\nu = 1$ correspond to the homogeneous translational flow, and $\alpha = \alpha(G_{ij})$, $\nu = 1/2$, to the arbitrary linear shear flow [equation (58)].

It is seen from equations (106) that at the Lewis–Semyonov number $Le_\infty = Pr_\infty/Sc_\infty = 1$ there is an analogy between the processes of mass and heat exchange of the particle with the flow.

Remark. In contrast to the case of an incompressible liquid (see Section 2.2) one cannot introduce into equations (105) the terms proportional to Pe^2 ln Pe, since in the case of a viscous heat conducting compressible gas the corresponding asymptotic expansion of the velocity field far from the particle is already unknown (except for the main term **i**, of course); for an incompressible liquid, such an expression is described by equation (9). Therefore, to obtain the next terms of the asymptotic expansion in small Reynolds number it is necessary already to investigate a full problem using, besides equations (74)–(79), also the equation of gas motion.

3.4. *The interpretation of terms: some remarks*

In a nonperturbed flow far from the particle (at $r \to \infty$), equations (97) and (100) 'loose memory' of the shape of the particle, which, in this region, can be interpreted as a point source. Therefore, to specify the structure of the asymptotic expansion for $\delta\Phi$ and δu at $r \to \infty$, consider first a very simple special case of a droplet or a particle of spherical shape ($r_s \equiv 1$), which will allow interpretation of the quantities $\bar{\Phi}$, $\bar{u}$ and $\delta\Phi$, δu.

Introduce the surface average operation [9, 16, 17]

$$\langle w \rangle = \frac{1}{4\pi r^2}\int_{\Sigma_r} w \,\mathrm{d}\Sigma_r = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} w(r,\eta,\phi)\,\mathrm{d}\phi\,\mathrm{d}\eta \quad (\eta = \cos\theta). \tag{107}$$

Note, that for any function w, depending only on the radial coordinate r, the equality $\langle w(r)\rangle = w(r)$ is valid and the mean-value operator [equation (107)] is permutable with the operator of differentiation over r. Note also, that in this case the solution is independent of the coordinate $\phi(\partial/\partial\phi = 0)$, while the zero terms of the inner expansions (89) depend, by virtue of the symmetry of the problems (12) and (88), only on r

$$\Phi_0 = J(\lambda)r^{-1} = \Phi_0(r), \quad u_0 = u_0(r).$$

Since equation (84) is valid for Φ and u, then for any analytical function f, one obtains, accurate to $o(Re_\infty)$, the following equation

$$\langle f(\Phi, u)\rangle = f(\langle\Phi\rangle, \langle u\rangle),$$

which is proved by direct verification.

Using the properties mentioned above and performing, similarly to refs. [16, 17], integration of equations and boundary conditions (93) and (94) over ϕ and η within the same limits as in equation (107), one obtains for the surface averages

$$\begin{gathered}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}}{\mathrm{d}r}\langle\Phi\rangle = 0;\\ r = 1,\quad \langle\Phi\rangle = J(\lambda);\quad r \to \infty,\quad \langle\Phi\rangle \to -\tfrac{1}{2}Pe_\mathrm{T}\,Nu(\lambda, 0),\end{gathered} \tag{108}$$

$$\begin{gathered}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\sigma(\langle\Phi\rangle)\frac{\mathrm{d}}{\mathrm{d}r}\langle u\rangle = 0;\\ r = 1,\quad \langle u\rangle = 1;\quad r \to \infty,\quad \langle u\rangle \to -\tfrac{1}{2}Pe_\mathrm{c}\,Sh(\sigma, 0).\end{gathered} \tag{109}$$

Equations (108) and (109) have been derived with the equalities $\langle\eta\rangle = 0$, $\langle\rho v_r\rangle_0 = 0$ taken into account, the latter being the corollary of the continuity equation (78) and of the condition of nonpenetrability through the particle surface, equation (4).

The mean Nusselt, equation (51), and Sherwood, equation (8), numbers are determined, accurate to $o(Re_\infty)$, by

$$Nu = -\left[\frac{\mathrm{d}\langle\Phi\rangle}{\mathrm{d}r}\right]_{r=1}, \quad Sh = -\left[\sigma(\langle\Phi\rangle)\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}r}\right]_{r=1}. \tag{110}$$

The comparison between the equations and boundary conditions (96), (99) and (108), (109) shows that in the case of a spherical particle one obtains the following equalities

$$\bar{\Phi} = \langle\Phi\rangle, \quad \delta\Phi = \Phi - \langle\Phi\rangle; \quad \bar{u} = \langle u\rangle; \quad \delta u = u - \langle u\rangle, \tag{111}$$

which means that $\bar{\Phi}$ and $\bar{u}$ are the surface mean quantities, while $\delta\Phi$ and δu are the deviations of the initial quantities Φ and u from their mean values. This, together with equation (110), shows that the second terms $\delta\Phi$ and δu [equations (95) and (98)] do not contribute to the integral inflows of heat and mass to the particle surface, i.e. $\delta Nu = \delta Sh = 0$. Having taken this into account and integrated equations (97) and (100) over the gas control volume V and then over the surfaces S and Σ_r, one obtains that the expansions of the functions $\delta\Phi$ and δu at $r \to \infty$ do not contain the 'source' term which would be proportional to r^{-1}.

The quantities $\delta\Phi$ and δu in the general case of arbitrarily shaped particles can be interpreted analogously. To this end, consider a set of surfaces $c_0 = \text{const.}$, where c_0 is the solution of problem (86) (in the case of a spherical particle this set, due to the equality $c_0 = r^{-1}$, consists of the surfaces of concentric spheres of constant radius, $r = \text{const.}$). It follows from equations (101) and (102) that the quantities $\bar{\Phi}$ and $\bar{u}$ will also be constant on these surfaces. The particle surface corresponds to $c_0 = 1$, while at a distance from the particle, at $r \to \infty$, the surfaces $c_0 = \text{const.}$ tend asymptotically to a spherical form due to equation (87). Far from the particle, equations (97) and (100) 'loose memory' of its shape (i.e. the structure of the solution at $r \to \infty$ will be the same as for a spherical particle), therefore the asymptotic expansion of the functions $\delta\Phi$ and δu at $r \to \infty$, just as in the case of a sphere, should not contain the 'source' terms proportional to r^{-1}. It follows (see Section 3.3) that the quantities $\delta\Phi$ and δu do not contribute to the mean Sherwood and Nusselt numbers, $\delta Nu = \delta Sh = 0$. This means that the functions $\delta\Phi$ and δu can be interpreted as fluctuations of the quantities Φ and u away from their average values, $\bar{\Phi}$ and $\bar{u}$, on the surfaces $c_0 = \text{const.}$

Remark 1. The first of equations (105) can be obtained directly from equation (93), just as it was done earlier in Section 2.2. For this, multiply both sides of equation (93) by Φ_0 and integrate over the control gas volume V with account for the first identity (23) and equality

$$\Phi_0(c_p\rho\mathbf{v}\ \text{grad}\ T)_0 = \Phi_0\ \text{div}\,(\rho\mathbf{v}n)_0 \equiv \text{div}\,(\rho\mathbf{v}\hat{E}) - \hat{E}\ \text{div}\,(\rho\mathbf{v}),$$

$$\hat{E} = \hat{E}(\Phi_0) = \int_0^{T_0(\Phi_0)} c_p(\xi)\Phi(\lambda,\xi)\,\mathrm{d}\xi,$$

the last term of which vanishes due to the continuity equation (78); here, the function $T_0(\Phi_0)$ is obtained by inversing the expression $\Phi_0 = \Phi(\lambda, T_0)$, equation (80). Then, going-over, by the Ostrogradsky–Gauss formula, to the surface integrals, one arrives at the expressions obtainable from equation (24) after the replacement of Pe by Pe_{T} and E by $\hat{E}$. Now, using equations (4), (87) and (89)–(91) and calculating these surface integrals by the procedure described in Section 2.2, one arrives at the first of equations (105).

Remark 2. One can consider a more general problem than equations (74)–(77), when the transport coefficients depend simultaneously on temperature and concentration

$$\lambda = \lambda(T,u), \quad \sigma = \sigma(T,u). \tag{112}$$

It can be shown in this case that the mean Nusselt and Sherwood numbers are assigned, accurate to $o(Re_\infty)$, by the expressions

$$\widehat{Nu} = J_{\mathrm{T}}\,Sh(1,0), \quad \widehat{Sh} = J_{\mathrm{c}}^{-1}\,Nu, \quad J_{\mathrm{T}} = \int_{-\varepsilon_{\mathrm{T}}}^{1} \lambda(T,u(T,J_{\mathrm{c}}))\,\mathrm{d}T. \tag{113}$$

Here, the function $u = u(T, J_{\mathrm{c}})$ is the solution of the following first-order ordinary differential equation depending on the parameter J_{c}

$$J_{\mathrm{c}}\frac{\mathrm{d}u}{\mathrm{d}T} = \frac{\lambda(T,u)}{\sigma(T,u)}, \quad u(1,J_{\mathrm{c}}) = 1, \tag{114}$$

while the parameter J_{c} can be found from the normalization-type condition

$$u(-\varepsilon_{\mathrm{T}}, J_{\mathrm{c}}) = -\varepsilon_{\mathrm{c}}; \quad \varepsilon_{\mathrm{T}} = \tfrac{1}{2}Re_\infty\,Pr_\infty\,\widehat{Nu}_0, \quad \varepsilon_{\mathrm{c}} = \tfrac{1}{2}Re_\infty\,Sc_\infty\,\widehat{Sh}_0, \tag{115}$$

where the quantities $\widehat{Nu}_0$ and $\widehat{Sh}_0$ are determined from equations (113)–(115) at $Re_\infty = 0$.

3.5. *Some examples: the case of the power law dependence of gas viscosity on temperature*

Consider now some of the specific examples. As usual one assumes that the conditions (79) have been fulfilled. Then, by virtue of equations (30), (79) and (106), the mean Nusselt and Sherwood numbers are prescribed by

$$\begin{aligned} Nu(\mu, Re_\infty) &= J(\mu)Sh(1,0) + \alpha\,Re_\infty^{\nu}\,Pr^{\nu}\,J(\mu)Sh^2(1,0) + o(Re_\infty), \\ Sh(\mu, Re_\infty) &= J(\mu)Sh(1,0) + \alpha\,Re_\infty^{\nu}\,J(\mu)Sh^2(1,0)\{Sc^{\nu}\,J(\mu) + Pr^{\nu}[1 - J(\mu)]\} + o(Re_\infty). \end{aligned} \tag{116}$$

In the specific case of the power dependence of gas viscosity on temperature

$$\mu = [1 + (T_{\mathrm{w}} - 1)T]^n, \quad T_{\mathrm{w}} = T_{\mathrm{s}}/T_\infty \quad (\mu_* = \text{const.}\cdot T_*^n),$$

one obtains for the integral $J(\mu)$

$$J(\mu) = \frac{1}{n+1}\,\frac{1 - T_{\mathrm{w}}^{n+1}}{1 - T_{\mathrm{w}}}. \tag{117}$$

For a spherical particle, $Sh(1,0) = 1$, in the case of a uniform translational flow, $\alpha = 1/2$, $\nu = 1$, equations (116) and (117), have been obtained in ref. [18] accurate to normalization and new notation. One can also find there a very interesting and practically important application of these results for the determination of the ion saturation current on an electric probe placed into a moving plasma (one of the methods of plasma diagnosis).

When the temperature depends on the gas viscosity, the Sutherland formula (C is the Sutherland constant depending on the kind of gas)

$$\mu = [1+(T_w-1)T]^{3/2}\left(1+\frac{T_w-1}{1+CT_\infty^{-1}}T\right)^{-1} \quad (\mu_* = \text{const.}\cdot T_*^{3/2}(T_*+C)^{-1})$$

yields the following expression for the integral $J(\mu)$ which defines equations (116)

$$J(\mu) = 2T_\infty^{-3/2}\left(\frac{C+T_\infty}{T_s-T_\infty}\right)\left\{\frac{1}{3}(T_s^{3/2}-T_\infty^{3/2})-C(T_s^{1/2}-T_\infty^{1/2})+C^{3/2}\left[\arctan\left(\frac{T_s}{C}\right)^{1/2}-\arctan\left(\frac{T_\infty}{C}\right)^{1/2}\right]\right\}. \tag{118}$$

Investigate now the effect of the gas viscosity on convective heat and mass transfer between a particle and a flow. The heat conduction and diffusion coefficients are assumed to be constant and independent of temperature

$$\lambda = D = 1, \quad \sigma = \rho. \tag{119}$$

It should be also taken into account that for small gas velocities the pressure in the flow differs slightly from the nonperturbed pressure at infinity (this difference has the squared Mach number order). This allows one in the state equation $p_* = \rho_* RT_*$ to replace the quantity p_* by $p_\infty = \rho_\infty RT_\infty$ and, after changing to the dimensionless variables, to obtain for the density the following expression

$$\rho = \rho(T) = [1+(T_w-1)T]^{-1}, \quad T_w = T_s/T_\infty. \tag{120}$$

In this case equations (105) retain their validity for the mean Nusselt and Sherwood numbers, where

$$J(\lambda) = 1, \quad J\left(\frac{\lambda}{\sigma}\right) = \frac{1}{2}(1+T_w). \tag{121}$$

Now, relate the resulting mean Sherwood number to an auxiliary Sherwood number, which corresponds to the case of an incompressible liquid

$$\frac{Sh(\sigma,0)|_{\sigma=\rho(T)}}{Sh(\sigma,0)|_{\sigma=1}} = \frac{2}{1+T_w}.$$

This equation shows that the gas compressibility leads to a lesser intensity of mass transfer to a 'hot' particle at $T_w > 1$, as compared to a similar process in an incompressible liquid, and increases the mass transfer in the case of a 'cold' particle at $T_w < 1$.

4. MASS AND HEAT TRANSFER BETWEEN A PARTICLE AND A FLOW IN THE PRESENCE OF VOLUMETRIC OR SURFACE CHEMICAL REACTIONS

4.1. *Stationary mass transfer of a particle in the presence of volumetric chemical reaction in the flow*

Consider the stationary problem of mass transfer of an arbitrarily shaped droplet in the case when the extractant, which is dissolved in the droplet, diffuses into the continuous phase and reacts there with the chemisorbent. If it is assumed that far from the particle the chemisorbent concentration is constant throughout the volume and that the flux of chemisorbent through the droplet surface is absent during the whole process, then for the stationary case this problem is described in dimensionless variables by the following equations and boundary conditions [22]

$$(\mathbf{v}\cdot\text{grad } c_e) = Pe^{-1}\,\Delta c_e - R(c_e, c_h), \tag{122}$$

$$(\mathbf{v}\cdot\text{grad } c_h) = s\,Pe^{-1}\,\Delta c_h - lR(c_e, c_h), \tag{123}$$

$$r = r_s(\theta,\phi), \quad c_e = 1, \quad \partial c_h/\partial n = 0, \tag{124}$$

$$r \to \infty, \quad c_e \to 0, \quad c_h \to 1, \tag{125}$$

$$c_e = \frac{c_{e*}}{c_{es}}, \quad c_h = \frac{c_{h*}}{c_{h\infty}}, \quad s = \frac{D_{h*}}{D_{e*}} = O(1), \quad l = \frac{c_{es}}{c_{h\infty}} = O(1),$$

$$R(c_e, c_h) \equiv aUc_{es}^{-1}R_*(c_{e*}, c_{h*}), \quad R(0, c_h) = R(c_e, 0) = 0; \quad Pe = aUD_{e*}^{-1}.$$

The volumetric reaction rate is assumed to have the following properties

$$R(c_e, c_h) = Pe\,H(c_e, c_h), \quad \max H = O(1), \quad 0 \leqslant c_e, \quad c_h \leqslant 1, \tag{126}$$

$$w = \frac{\partial H}{\partial c_e}(0,1) = O(1), \quad H(0,1) = 0. \tag{127}$$

Condition (126) means that the reaction is weak, $R = O(Pe)$, while condition (127) shows that far from the particle the reaction follows the first order of the extractant. In particular, condition (127) is satisfied by the $n+1$-order reaction which follows the first order for the extractant and the nth order for the chemisorbent

$$H(c_e, c_h) = w c_e c_h^{n+1}, \quad w = k_v D_{e*} U^{-2} c_{h\infty}^{n} = O(1). \tag{128}$$

Here $n = 0$ corresponds to the first-order chemical reaction [in this case the first equation (122) is solved independently of the second equation (123)], while $n = 1$ corresponds to the second-order reaction, k_v is the volumetric reaction rate constant.

Investigate the problem (122)–(127) by the method of matched asymptotic expansions in small Péclet number. For the velocity field of the incompressible liquid one uses equations (3), (4) and (9) corresponding to the translational Stokes flow past a particle. The main parameter sought in this problem is the mean Sherwood number for the first reacting component (extractant). This number is determined from equation (8) at $D = 1$ and $c = c_e$.

For c_e, the terms of the inner expansion in the region Ω_1 can be presented in the form of equation (10) at $c_e \equiv \Phi$, $c_{ei} \equiv \Phi_i$ $(i = 0, 1, 2)$ and these satisfy the equations and boundary conditions (12)–(14). The first term of the outer expansion (as earlier, the zero term is equal to zero) is described, by virtue of equations (126) and (127), by

$$(\mathbf{i} \cdot \operatorname{grad}_z c_e^{(1)}) = \Delta_z c_e^{(1)} - w c_e^{(1)}; \quad z \to \infty, \quad c_e^{(1)} \to 0, \tag{129}$$

the solution of which, satisfying the condition of matching with the zero term of the inner expansion (15), is of the form [see equations (32) and (40)]

$$c_e^{(1)} = \frac{q}{z} \exp\left\{\frac{z}{2}\left[\eta - (1+4w)^{1/2}\right]\right\}, \quad q = Sh(1,0). \tag{130}$$

It can be shown that at $z \to 0$ the next term of the outer expansion generates the same logarithmic singularity which appeared in the problems of Section 2 and altered the power-law character of the inner expansion. Taking this into account and using equation (130), one obtains that the concentration c_e on the general boundary between the outer, Ω_∞, and the inner, Ω_1, regions can be presented in the form of equation (21), where

$$c_e = \Phi, \quad \beta(Pe) = -\tfrac{1}{2} q\left[Pe(1+4w)^{1/2} + Pe^2 \ln Pe(\mathbf{F} \cdot \mathbf{i})\right]. \tag{131}$$

Omitting the intermediate calculations, which are similar to those of Section 2.2, one finally obtains for the mean Sherwood number

$$Sh = Sh(1,0) + \tfrac{1}{2} Pe(1+4w)^{1/2} Sh^2(1,0) + \tfrac{1}{2} Pe^2 \ln Pe(\mathbf{F} \cdot \mathbf{i}) Sh^2(1,0) + O(Pe^2). \tag{132}$$

It is seen that when there is no reaction, $R = 0$ $(w = 0)$, equation (132) passes into equation (29).

4.2. *Nonstationary mass transfer of a particle in the presence of a volumetric reaction*

Consider now a nonstationary analogue of the problem (122)–(127). The corresponding equations and initial conditions for the case of a sudden occurrence of chemical reaction are of the form

$$\frac{\partial c_e}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} c_e) = Pe^{-1} \Delta c_e - R(c_e, c_h); \quad t = 0, \quad c_e = 0, \tag{133}$$

$$\frac{\partial c_h}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} c_h) = s\, Pe^{-1} \Delta c_h - lR(c_e, c_h); \quad t = 0, \quad c_h = 1. \tag{134}$$

The equations and boundary conditions (133), (134) should be supplemented by the boundary conditions (123), (124); it is also assumed that the rate of the volumetric chemical reaction satisfies equations (126) and (127).

The asymptotic behaviour of problem (133), (134), (124)–(127) is investigated at large times. The condition (41) is assumed to be fulfilled. Then all the terms of the inner expansion are described by the stationary equations (12)–(14) at $c_{ei} \equiv \Phi_i$, the solution of which depends parametrically on temperature due to the boundary condition at infinity (the condition of matching with the outer expansion). After passing to the transform by using the Laplace–Carlson transformation, the first term of the outer expansion will satisfy equation (129) with the additional term $p\, Pe^{-1} c_e^{(1)}$ on the LHS [see equation (39)]. The solution of this equation, which at $t \to \infty$ $(p \to 0)$ becomes stationary, is determined by equation (130), with w replaced by $w + p\, Pe^{-1}$.

By using the same sort of reasoning as with the stationary case, one arrives at the concentration distribution on the general boundary between the inner and outer regions. This distribution is prescribed by equation (21), in which the parameter $\beta(Pe)$ has been determined from equation (131) with w replaced by $w + p\, Pe^{-1}$. Inversion to the original function and appropriate calculations yield the following equation for the mean Sherwood number

$$Sh(t) = Sh(1,0) + \tfrac{1}{2} Pe(1+4w)^{1/2} g(et) Sh^2(1,0) + \tfrac{1}{2} Pe^2 \ln Pe(\mathbf{F} \cdot \mathbf{i}) Sh^2(1,0) + O(Pe^2), \quad e = (w + \tfrac{1}{4}) Pe. \tag{135}$$

At $t \to \infty$, equation (135) passes into equation (132), while at $w \to 0$, it passes into equation (44).

Note, that the derivation of equations (132) and (135) has not required the knowledge of the chemisorbent concentration distribution in the flow; it was sufficient to know the asymptotic behaviour of the quantity c_h at infinity ($r \to \infty$, $c_h \to 1$).

4.3. *Nonstationary mass transfer of a particle in the presence of the surface first-order chemical reaction*

Investiate the development of the stationary mass transfer of an arbitrarily shaped droplet or solid particle in the presence of the first-order chemical reaction on its surface. In dimensionless variables, the associated boundary-value problem has the form

$$\frac{\partial c}{\partial t}+(\mathbf{v}\cdot \operatorname{grad}\, c)=Pe^{-1}\,\Delta c; \quad t=0, \quad c=0, \tag{136}$$

$$r=r_s, \quad \frac{\partial c}{\partial n}=k(c-1); \quad r\to\infty, \quad c\to 0, \tag{137}$$

$$c=\frac{c_\infty-c_*}{c_\infty}, \quad k=\frac{ak_s}{D_*}.$$

Here k_s is the rate constant of the surface chemical reaction. No restrictions are placed on the dimensionless chemical reaction rate constant k.

For the liquid velocity field equations (3), (4) and (9) are used. The approximation for the asymptote to the solution of the problem, equations (136) and (137), is investigated at large times, equation (41).

All the terms of the inner expansion (10) at $c \equiv \Phi$, $c_i \equiv \Phi_i(t, r, \theta, \phi)$ ($i = 0, 1, 2$) satisfy the stationary equations (12)–(14) subject to the following boundary conditions on the particle surface

$$r=r_s, \quad \frac{\partial c_0}{\partial n}=k(c_0-1), \quad \frac{\partial c_1}{\partial n}=kc_1, \quad \frac{\partial c_2}{\partial n}=kc_2. \tag{138}$$

Therefore, by virtue of the conditions of matching with the outer expansion, all the terms of the inner expansion depend on the time only parametrically; moreover, due to the equality $c^{(0)}=0$, the zero term of the inner expansion, c_0, is independent of time and is prescribed by equation (15) [12].

The outer terms of expansion (11) are described by the same nonstationary equations as in the case of a purely diffusive reaction regime at $r = r_s$, $c_0 = 1$ [which corresponds to the limiting transition $k \to \infty$ in the boundary condition (137)]. In particular, the first term of the outer expansion is determined, at $c^{(1)} \equiv \Phi^{(1)}$, by equation (38), the solution transform of which is prescribed by equation (40). Reasoning as before, one arrives at equation (42) at $c \equiv \Phi$ for the concentration on the general boundary between the outer and inner regions.

Then, multiplying the equations for the inner terms of the expansion (12)–(14), respectively by c_0, $Pe\, c_0$, $Pe^2 \ln Pe\, c_0$, one comes to equality (22), where $c \equiv \Phi, c_0 \equiv \Phi_0 = c(\Phi_0)$. Taking into account the identities (23) (in this case $E = c_0^2/2$) and integrating equation (22) over the control liquid volume V, one obtains expression (24). The last four integrals I_j ($j = 3, 4, 5, 6$) in equation (24) are calculated, as earlier, with equalities (4), (15), and (42) taken into account. The sum of the first two integrals is transformed by substituting, in place of the first factors, their values on the particle surface by the formulae

$$r=r_s, \quad c=1+\frac{1}{k}\frac{\partial c}{\partial n}, \quad c_0=1+\frac{1}{k}\frac{\partial c_0}{\partial n} \quad (c=c_0+Pe\, c_1+Pe^2 \ln Pe\, c_2),$$

which result from the boundary conditions (137) and (138). This yields

$$I_1+I_2=-\int_S c_0\frac{\partial c}{\partial n}\,dS+\int_S c\frac{\partial c_0}{\partial n}\,dS=-\int_S\frac{\partial c}{\partial n}\,dS+\int_S\frac{\partial c_0}{\partial n}\,dS=4\pi(Sh-Sh_0).$$

Having performed some calculations, one finally obtains

$$Sh=Sh_0+\tfrac{1}{2}Sh_0^2[Pe\, g(\tau)+Pe^2 \ln Pe(\mathbf{F}\cdot\mathbf{i})]+O(Pe^2). \tag{139}$$

Here, $Sh_0 = q$ is the mean Sherwood number corresponding to the mass transfer of a particle with a stationary medium ($Pe = 0$) in the presence of the first-order chemical reaction (137) on its surface. In particular, for a spherical droplet or a solid particle

$$Sh_0=\frac{k}{k+1}. \tag{140}$$

Note that equation (139) extends the results of ref. [11] to the nonstationary case.

Remark. It can be shown that equation (139) subject to condition (41) is valid also for a more complex boundary-value problem which is described by the equation and initial condition (136) with the nonstationary boundary condition on the (solid) particle surface

$$r = r_s, \quad \Xi\frac{\partial c}{\partial t} = \frac{\partial c}{\partial n} - k(c-1); \quad r \to \infty, \quad c \to 0,$$
$$\Xi = U\Gamma D_*^{-1}, \quad k = ak_{*s}\Gamma D^{-1} \quad (\Xi = O(1)). \tag{141}$$

This equation is written down presuming the adsorptional equilibrium and was obtained from the following dimensional equations on the particle surface

$$\frac{\partial c_s}{\partial t_*} = D\frac{\partial c_*}{\partial n_*} - k_{*s}c_s, \quad c_s = \Gamma c_* \quad (r = r_s).$$

In the stationary case (136) ($\partial/\partial t = 0$) of an arbitrarily shaped particle (droplet) in a shear flow, equation (57), the following formula is valid for the first-order chemical reaction on its surface, equation (137)

$$Sh = Sh_0(1-\alpha\, Pe^{\nu}\, Sh_0)^{-1} + O(Pe), \quad \nu = (n+1)^{-1},$$

which is similar to the expression for the mean Sherwood number, equation (67). It can be shown that for the linear shear flow, equation (58) ($n = 1$) the accuracy of this formula is still higher and has the order of $O(Pe^{3/2})$.

4.4. *Nonstationary nonisothermal chemical reaction on the surface of a spherical particle*

Consider the nonstationary multi-component diffusion to a reacting spherical droplet or a solid particle in a translational Stokes flow in the presence of a nonisothermal chemical reaction on the particle surface, the rate of which depends arbitrarily on temperature and concentration. It is assumed that the reaction is quasi-stationary, the particle is non-conducting, the concentration of the reacting species are rather weak so that the surface reaction does not influence the parameters of the flow and of the particle.

The dimensionless equations of convective diffusion and heat conduction, the initial and boundary conditions that express the uniformity of temperature and concentrations far from the particle, the 'reaction law' and the heat balance on its surface are of the form [24]

$$\frac{\partial c_m}{\partial t} + (\mathbf{v}\cdot\text{grad } c_m) = Pe_m^{-1}\,\Delta c_m, \quad (1 < r < \infty) m = 1,\ldots,M, \tag{142}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\text{grad } T) = Pe_0^{-1}\,\Delta T, \quad (1 < r < \infty), \tag{143}$$

$$t = 0, \quad c_m = 0, \quad T = 0, \tag{144}$$

$$r \to \infty, \quad c_m \to 0, \quad T \to 0, \tag{145}$$

$$r = 1, \quad -\frac{\partial c_m}{\partial r} = f_m(T, c_1,\ldots,c_M), \tag{146}$$

$$r = 1, \quad -\frac{\partial T}{\partial r} = \sum_{m=1}^{M} h_m f_m(T, c_1,\ldots,c_M), \tag{147}$$

$$\psi_{\text{st}} = \frac{1}{2}\left(r^2 - \frac{1}{2}\frac{3\beta+2}{\beta+1} + \frac{1}{2}\frac{\beta}{\beta+1}\frac{1}{r}\right)\sin^2\theta, \quad (\mathbf{v}\cdot\text{grad } c) = \frac{1}{r^2\sin\theta}\frac{\partial(c,\psi_{\text{st}})}{\partial(r,\theta)}, \tag{148}$$

$$c_{m*} = c_{m\infty}(1-c_m), \quad T_* = T_\infty(1-T), \quad Pe_m = aUD_{m*}^{-1},$$
$$Pe_0 \equiv Pe_{\text{T}} = aUc_{p*}\rho_*\lambda_*^{-1}, \quad h_m = c_{m\infty}H_mD_{m*}(\lambda_*T_\infty)^{-1},$$
$$f_m(T, c_1,\ldots,c_M) \equiv a(c_{m\infty}D_{m*})^{-1}f_{m*}(T_*, c_{1*},\ldots,c_{M*}).$$

Here it is assumed for simplicity that $T_\infty, c_{m\infty} \neq 0$ ($m = 1,\ldots,M$).

Investigate the boundary-value problem (142)–(148) by the method of matched asymptotic expansions in small Péclet number

$$Pe \to 0, \quad Pe_m = Pe\,\Theta_m, \quad \Theta_m = O(1); \quad m = 0,1,\ldots,M \tag{149}$$

on the assumption that condition (41) is fulfilled.

In the inner region, $\Omega_1 = \{1 \leqslant r \leqslant O(Pe^{-1})\}$, the distribution of temperature and concentrations is sought, accurate to $O(Pe^2)$, in the form [16, 17]

$$T = T_0 + Pe_0\, T_1 + Pe_0^2 \ln Pe_0\, T_2, \quad c_m = c_{m0} + Pe_m\, c_{m1} + Pe_m^2 \ln Pe_m\, c_{m2},$$
$$T_0 = q_0 r^{-1}, \quad c_{m0} = q_m r^{-1} \quad (q_0, q_m = \text{const.}; m = 1,\ldots,M). \tag{150}$$

The substitution of this equation into equations (142) and (143) with the subsequent separation of the terms with the same 'powers' of the small parameter Pe and with regard for equation (41) leads to the following stationary equations

$$\Delta T_0 = 0, \quad \Delta T_1 = q_0 r^{-4} \partial\psi_{\mathrm{st}}/\partial\eta, \quad \Delta T_2 = 0 \quad (\eta = \cos\theta), \tag{151}$$

$$\Delta c_{m0} = 0, \quad \Delta c_{m1} = q_m r^{-4} \partial\psi_{\mathrm{st}}/\partial\eta, \quad \Delta c_{m2} = 0. \tag{152}$$

The first terms of the outer expansion ($T^{(0)} = c_m^{(0)} = 0$), after the substitution of the auxiliary variables $z_m = Pe_m r$, will be described by the linear nonstationary equations (38) with the corresponding replacement $Pe \to Pe_m$, $z \to z_m$. The Laplace transform of the sôlution of these equations is as given by equation (40). Taking into account that the second terms of the outer expansion have a logarithmic singularity at $z_m \to 0$ and reasoning as before, one obtains, accurate to $O(Pe^2)$, the following equations for the distribution of temperature and concentrations on the general boundary between the inner, Ω_1, and outer, Ω_∞, regions

$$T = \Psi_0(Pe) r^{-1} - \tfrac{1}{2} Pe_0\, q_0 \left\{ g(\tau_0)(1-\eta) + Pe_0 \ln Pe_0 \left(\frac{3\beta+2}{3\beta+3} \right) \right\}, \tag{153}$$

$$c_m = \Psi_m(Pe) r^{-1} - \tfrac{1}{2} Pe_m\, q_m \left\{ g(\tau_m)(1-\eta) + Pe_m \ln Pe_m \left(\frac{3\beta+2}{3\beta+3} \right) \right\}, \tag{154}$$

$$\begin{gathered} \Psi_m(Pe) \equiv \Psi_m(Pe_0, Pe_1, \ldots, Pe_M; t) = q_m + A_m^{(i)}(\tau_i) Pe_i + B_m^{(i)}\, Pe_i^2 \ln Pe_i, \\ \tau_m = \tfrac{1}{4} Pe_m\, t \quad (m = 0, 1, \ldots, M;\ B_m^{(i)} = \text{const.}). \end{gathered} \tag{155}$$

Here, the summation is taken over the subscript i ($i = 0, 1, \ldots, M$); the functions Ψ_0, Ψ_m and the parameters q_0, q_m should be determined in the course of the problem solution.

Now, using equation (109), one introduces the surface mean-value operation, which in the given axisymmetric case is determined as

$$\langle w \rangle = \frac{1}{2} \int_{-1}^{1} w(r, \eta)\, \mathrm{d}\eta \quad (\eta = \cos\theta). \tag{156}$$

With equation (156) taken into account, the mean Sherwood and Nusselt numbers, equation (8), can be written down in the form

$$Sh_m = -\left\langle \frac{\partial c_m}{\partial r} \right\rangle \bigg|_{r=1}, \quad Nu = -\left\langle \frac{\partial T}{\partial r} \right\rangle \bigg|_{r=1}. \tag{157}$$

Multiply equations (151) successively by 1, Pe_0, $Pe_0^2 \ln Pe_0$ [or by 1, Pe_m, $Pe_m^2 \ln Pe_m$ for equation (152)] and make the addition; integrating further over $\eta \in [-1, 1]$ gives

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}}{\mathrm{d}r} \langle c_m \rangle = 0, \quad \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}}{\mathrm{d}r} \langle T \rangle = 0. \tag{158}$$

The boundary conditions at $r \to \infty$ (the conditions of matching) for equations (158) are obtained by averaging equations (153)–(155) with regard for the equality $\langle \eta \rangle = 0$. Subject to these conditions, the solution of equations (158) is found in the form

$$\begin{gathered} \langle T \rangle = \Psi_0(Pe) \left\{ r^{-1} - \tfrac{1}{2} Pe_0\, g(\tau_0) - \tfrac{1}{2} Pe_0^2 \ln Pe_0 \left(\frac{3\beta+2}{3\beta+3} \right) \right\} + O(Pe^2), \\ \langle c_m \rangle = \Psi_m(Pe) \left\{ r^{-1} - \tfrac{1}{2} Pe_m\, g(\tau_m) - \tfrac{1}{2} Pe_m^2 \ln Pe_m \left(\frac{3\beta+2}{3\beta+3} \right) \right\} + O(Pe^2). \end{gathered} \tag{159}$$

When the above expressions were derived it was taken into account that the parameters q_m ($m = 0, 1, \ldots, M$) can be replaced, without the loss of accuracy, by the functions $\Psi_m(Pe)$, equation (155), for the concentrations and temperatures on the general boundary between the inner and outer regions, equations (153) and (154).

Since c_{m0} and T_0 depend only on the radial coordinate r, equation (150), then, by virtue of equation (150), the following formula is valid, accurate to $O(Pe^2)$, for any analytical function f

$$\langle f(T, c_1, \ldots, c_M) \rangle = f(\langle T \rangle, \langle c_1 \rangle, \ldots, \langle c_M \rangle), \tag{160}$$

which is proved by direct verification with regard for the properties of the mean-value operation, equation (156).

The comparison of equations (157) and (158) yields the equalities $Nu = \Psi_0(Pe)$, $Sh_m = \Psi_m(Pe)$. Taking these into account and averaging the boundary conditions, equations (146) and (147), and then using equations (159), the following system of algebraic (transcendental) equations is obtained

$$
\begin{aligned}
Sh_m &= f_m\left(\frac{Nu}{Nu_\infty}, \frac{Sh_1}{Sh_{1\infty}}, \ldots, \frac{Sh_M}{Sh_{M\infty}}\right); \quad m = 1, \ldots, M, \\
Nu &= \sum_{m=1}^{M} h_m Sh_m,
\end{aligned}
\tag{161}
$$

for the determination of the mean Sherwood and Nusselt numbers. Here, the last equation is obtained by the substitution of the expressions for f_m from equation (146) into equation (147), while the quantities $Sh_{m\infty} \equiv Sh(1, Pe_m), Nu_\infty \equiv Sh(1, Pe_0)$ correspond to the purely diffusional (thermal) reaction regime on the sphere surface ($r = 1, c_m = 1, T = 1$) and are of the form

$$
\begin{aligned}
Sh_{m\infty} &= 1 + \tfrac{1}{2} Pe_m\, g(\tau_m) + \tfrac{1}{2} Pe_m^2 \ln Pe_m \left(\frac{3\beta+2}{3\beta+3}\right) + O(Pe^2), \\
Nu_\infty &= 1 + \tfrac{1}{2} Pe_0\, g(\tau_0) + \tfrac{1}{2} Pe_0^2 \ln Pe_0 \left(\frac{3\beta+2}{3\beta+3}\right) + O(Pe^2).
\end{aligned}
\tag{162}
$$

Equations (161) and (162) are the extensions of the results of refs. [16, 17] to the nonstationary case.

It can be shown that equations (161) and (162), subject to condition (41), are also valid for a more complex boundary-value problem, equations (142)–(145), with nonstationary analogues of the boundary conditions on the particle surface, equations (146) and (141) (to the LHS of which the terms $\Xi_m \partial c_m/\partial t$, $\Xi_0 \partial T/\partial t$ are added; see also 'Remark' at the end of Section 4.3).

Remark. In the stationary case of a spherical particle in an arbitrary shear flow, equation (57), the algebraic equations (161) held their validity for the mean Sherwood and Nusselt numbers. The parameters $Sh_{m\infty}$ and Nu_∞ in these equations are determined by equation (67) ($Sh(1, 0) = 1$, $Pe \to Pe_m$, $n \to n+1$) and have the form

$$
Sh_{m\infty} = \sum_{i=0}^{n+1} \alpha^i\, Pe_m^{\nu i} + O(Pe^{1+\nu}), \quad Nu_\infty = \sum_{i=0}^{n+1} \alpha^i\, Pe_0 + O(Pe^{1+\nu}). \tag{163}
$$

The comparison of equations (161), (162) and (161), (163) shows that at small Péclet numbers, equations (149), the system (161) provides, in all of the cases, the correct asymptotic result accurate to $o(Pe)$.

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UNE ANALYSE ASYMPTOTIQUE DE QUELQUES PROBLEMES, A VALEUR LIMITE NON LINEAIRE, DE CONVECTION MASSIQUE ET THERMIQUE ENTRE DES PARTICULES REACTIVES ET L'ECOULEMENT

Résumé—La première partie de ce texte considère le transfert de chaleur et de masse d'une particule solide de forme quelconque dans des écoulements, de translation et de cisaillement, d'un fluide incompressible dans le cas où le coefficient de conduction thermique (diffusion) est une fonction arbitraire de la température (concentration). Pour l'écoulement de translation, les trois premiers termes du développement asymptotique du nombre de Nusselt moyen sont obtenus en fonction du petit nombre de Péclet, et deux termes pour l'écoulement de cisaillement. Les particules étudiées ont la forme d'une sphère, d'un disque, d'un ellipsoïde ou d'haltères. La seconde partie du texte considère un échange simultané de chaleur et de masse d'une particule de forme arbitraire avec l'écoulement de translation d'un gaz comprimé dont les paramètres (les coefficients de conduction thermique et de diffusion et les chaleurs massiques du gaz) dépendent arbitrairement de la température. Pour les nombres de Nusselt et de Sherwood moyens, on détermine les deux premiers termes du développement asymptotique en fonction du nombre de Reynolds qui est faible. La troisième partie considère le problème de transfert massique d'une goutte dans le cas où un extractant dissout diffuse dans une phase continue et entre en réaction du second ordre avec un chimisorbant. Un développement asymptotique à trois termes pour le nombre de Sherwood moyen est obtenu en fonction du nombre de Péclet. On étudie la diffusion variable à une particule réactante dans un écoulement laminaire de translation, dans le cas où une réaction chimique non isotherme se produit sur la surface, la vitesse dépendant arbitrairement de la température et des concentrations.

ASYMPTOTISCHE UNTERSUCHUNG EINIGER NICHTLINEARER RANDWERTPROBLEME BEIM KONVEKTIVEN STOFF- UND WÄRMEÜBERGANG ZWISCHEN REAGIERENDEN PARTIKELN UND EINER STRÖMUNG

Zusammenfassung—Der erste Teil dieser Arbeit, Abschnitt 2, befaßt sich mit dem Stoff- und Wärmetransport an einem Tröpfchen oder festen Partikel von beliebiger Form, das sich in der Translations- und Scherströmung eines inkompressiblen Fluids befindet, wobei der Wärmeleit-(Diffusions-)Koeffizient eine beliebige Funktion der Temperatur (Konzentration) ist. Bei der Translationsströmung ergeben sich die ersten drei Glieder einer asymptotischen Reihenentwicklung für die mittlere Nusselt-Zahl als Funktion niedriger Peclet-Zahlen, bei der Scherströmung die ersten beiden Glieder. Die untersuchten Partikel hatten Kugel-, Scheiben-, Ellipsoid- und Hantel-Form. Der zweite Teil dieser Arbeit, Abschnitt, 3, befaßt sich mit dem gleichzeitigen Wärme- und Stoffaustausch an einem Partikel von beliebiger Form, das sich in der Translationsströmung eines komprimierten Gases befindet, dessen wesentliche Parameter (die Wärmeleit- und Diffusionskoeffizienten sowie die spezifische Wärmekapazität des Gases) auf beliebige Weise von der Temperatur abhängen. Für die mittlere Nusselt- und Sherwood-Zahl werden die ersten beiden Glieder der entsprechenden asymptotischen Reihenentwicklung als Funktion niedrigerer Reynolds-Zahlen bestimmt. Der dritte Teil dieser Arbeit, Abschnitt 4, befaßt sich mit dem Stofftransport an einem Tröpfchen, der dann auftritt, wenn ein darin gelöstes Extraktionsmittel in eine kontinuierliche Phase hineindiffundiert und dort in eine chemische Reaktion zweiter Ordnung mit einem chemischen Absorptionsmittel eintritt. Für die mittlere Sherwood-Zahl ergibt sich eine asymptotische Reihenentwicklung mit drei Gliedern als Funktion der Peclet-Zahl. Die instationäre Diffusion an einem reagierenden Partikel wurde in einer laminaren Translationsströmung für den Fall einer nichtisothermen chemischen Reaktion an dessen Oberfläche untersucht. Der Diffusionsmassenstrom hängt dabei wesentlich von Temperatur und Konzentration ab.

АСИМПТОТИЧЕСКИЙ АНАЛИЗ НЕКОТОРЫХ НЕЛИНЕЙНЫХ КРАЕВЫХ ЗАДАЧ КОНВЕКТИВНОГО МАССОТЕПЛООБМЕНА РЕАГИРУЮЩИХ ЧАСТИЦ С ПОТОКОМ

Аннотация—В первой части работы в разделе 2 рассматривается массо- и теплообмен капли или твердой частицы любой формы в поступательном и сдвиговом потоках несжимаемой жидкости и газа в случае, когда коэффициент теплопроводности (диффузии) произвольным образом зависит от температуры (концентрации). В случае поступательного потока получены три, а в случае сдвигового — два первых члена асимптотического разложения среднего числа Нуссельта по малому числу Пекле. Рассмотрены случаи сферической, дискообразной, эллипсоидальной и гантелевидной формы частиц. Во второй части работы в разделе 3 исследован совместный тепломассообмен частицы любой формы с поступательным потоком сжимаемого газа, определяющие параметры которого (коэффициенты теплопроводности и диффузии, а также удельная теплоемкость газа) произвольным образом зависят от температуры. Для средних чисел Нуссельта и Шервуда определены первые два члена соответствующего асимптотического разложения по малому числу Рейнольдса. В третьей части работы в разделе 4 рассматривается задача о массообмене капли в случае, когда растворенный в капле экстрагент, диффундируя в сплошную фазу, вступает там в химическую реакцию второго порядка с хемосорбентом. Для среднего числа Шервуда получено трехчленное асимптотическое разложение по числу Пекле. Исследуется нестационарная диффузия к реагирующей сфере, обтекаемой ламинарным поступательным потоком при протекании на ее поверхности неизотермической химической реакции, скорость которой произвольным образом зависит от температуры и концентраций.